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Hamilton Cycles in Euler Tour Graph

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In this paper we define the Euler tour graph of an Eulerian graph by K-transformations, which was introduced by Kotzig in 1966 (*in* "Theory of Graphs" (P. Erdös and G. Katona, Eds.), Proc., Colloq., Tihany, Hungary, September, 1966, Akad. Kaido, Hungarian Academy of Sciences, Budapest, 1968) and prove that any edge in an Euler tour graph is in a Hamilton cycle. © 1986 Academic Press, Inc.

Let G be an Eulerian graph without loops, which may have multiple edges, and let E be an Euler tour of G [4]. For every vertex v of G with $\deg v = 2t \geqslant 4$, E passes through v exactly t times. So we may write E: $e_0 v e_1 \cdots e_2 v e_3 \cdots e_i v e_{i+1} \cdots e_{2t-2} v e_{2t-1} \cdots e_0$. A triple (e_i, v, e_{i+1}) is called a transition of E through v. Two Euler tours with opposite directions will be regarded as the same, and so do the transitions. A subsequence of E starting from and ending at v which contains at least one edge is called a v–v segment of E. An Euler tour F is said to be obtained from E by a K-transformation at v on a segment S if F is obtained from E by changing the direction of travel along S [1], [2]. The Euler tour graph of G, denoted by $\mathrm{Eu}(G)$, is the undirected simple graph defined as follows: The vertices of $\mathrm{Eu}(G)$ are the Euler tours of G, and two Euler tours E and F are adjacent in $\mathrm{Eu}(G)$ if they can be transformed from each other by a K-transformation.

The first author introduced the concept of the Euler tour graph in 1981 (on a seminar in Lanzhou University) and conjectured that any Euler tour graph is Hamiltonian. Xia [5] proved that the conjecture is true for any graph with maximum degree at most four. In this paper, we prove that the conjecture is true in general. In fact, we have the following stronger result:

THEOREM. *Let G be an Eulerian graph having at least three Euler tours. Then* $\mathrm{Eu}(G)$ *is edge-Hamiltonian* (*i.e., any edge of* $\mathrm{Eu}(G)$ *is contained in a Hamilton cycle of* $\mathrm{Eu}(G)$).

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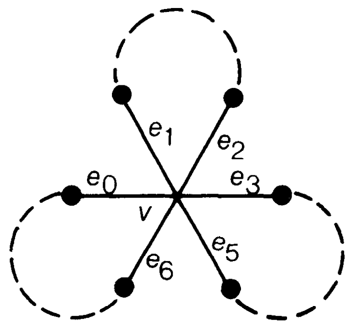

FIGURE 1

Proof. Let Q be the set of vertices of degree at least four in G, and let λ be the sum of degrees of vertices in Q. The proof is by induction on λ. Since G has at least three Euler tours, $\lambda \geqslant 6$. If $\lambda = 6$, then G is the graph shown in Fig. 1. It is easy to see that G has precisely eight Euler tours and $\mathrm{Eu}(G)$ is as shown in Fig. 2. The conclusion is evident.

Now suppose that the conclusion is true for any graph with $\lambda = 2m$, where m is an integer at least three. Let $\lambda = 2m + 2$. Take an edge $E_1 E_2$ of $\mathrm{Eu}(G)$, where E_1 and E_2 are Euler tours of G. By definition, E_2 is obtained from E_1 by a K-transformation at a vertex v of G. Let $E_1 = e_0 v e_1 \cdots e_2 v e_3 \cdots e_i v e_{i+1} \cdots e_0$, and $E_2 = e_0 v e_i \cdots e_{i-1} v e_{i-2} \cdots e_1 v e_{i+1} \cdots e_0$. Denote the set of Euler tours of G containing the transition (e_0, v, e_j) by S_j, $1 \leqslant j \leqslant 2t - 1$. Then it is obvious that $S_1, S_2, \ldots,$ form a partition of the vertex set of $\mathrm{Eu}(G)$, and $S_j = \emptyset$ if and only if $\{e_0, e_j\}$ is an edge cut of G and otherwise $|S_j| \geqslant 2$. Let K_j be the subgraph of $\mathrm{Eu}(G)$ induced by S_j. Since K_j is isomorphic to the Euler tour graph of a graph obtained from G by substituting two vertices v' and v'' for v such that e_0 and e_j are incident to v'

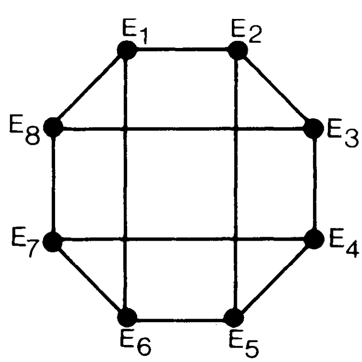

FIGURE 2

and the other edges incident to v in G are incident to v'', by the induction hypothesis, K_j is edge Hamiltonian if $|S_j| \geqslant 3$.

Now we are going to find a cycle C in Eu(G) satisfying the following conditions:

(i) the length of C is even;

(ii) C contains E_1E_2;

(iii) for each j such that $S_j \neq \varnothing$, C has exactly one edge a_j in K_j, and

(iv) the edges a_1, a_2,..., form a perfect matching of C.

If there exists such a cycle C in Eu(G), denoting H_j a Hamilton cycle containing the edge a_j in K_j (if $|S_j| = 2$, let $H_j = \varnothing$), then $(H_1 \cup H_2 \cup \cdots \cup H_j \cup \cdots)\,\Delta C$ is a Hamilton cycle of Eu(G) containing E_1E_2 (shown in Fig. 3), thereby the theorem is proved.

We consider the following three cases:

Case 1. $\deg v = 4$. Without loss of generality, we may assume that $E_1 = e_0ve_1 \cdots e_2ve_3 \cdots e_0$ and $E_2 = e_0ve_2 \cdots e_1ve_3 \cdots e_0$. Two subcases should be considered:

(1) Suppose that v is a cut vertex of G (see Fig. 4). In this case, $V(\mathrm{Eu}(G)) = S_1 \cup S_2$, since $\lambda > 6$, there is a vertex u with degree at least four in segment $ve_1 \cdots e_2v$ (or $ve_3 \cdots e_0v$). Then the required cycle $C = F_1F_2F_3F_4F_1$ is as follows:

$$F_1 = e_0ve_1 \cdots ue_1' \cdots e_2'u \cdots e_2ve_3 \cdots e_0 = E_1$$
$$F_2 = e_0ve_2 \cdots ue_2' \cdots e_1'u \cdots e_1ve_3 \cdots e_0 = E_2$$
$$F_3 = e_0ve_2 \cdots ue_1' \cdots e_2'u \cdots e_1ve_3 \cdots e_0,$$
$$F_4 = e_0ve_1 \cdots ue_2' \cdots e_1'u \cdots e_2ve_3 \cdots e_0.$$

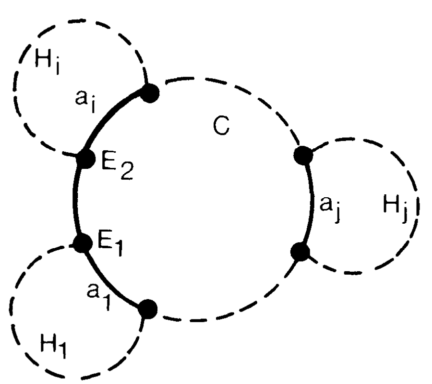

FIGURE 3

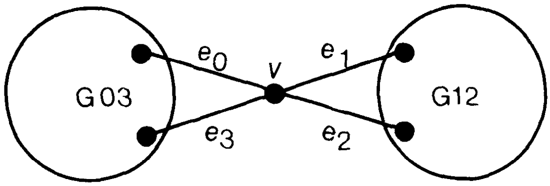

FIGURE 4

(2) Suppose that v is not a cut vertex of G (see Fig. 5). Then $V(\mathrm{Eu}(G)) = S_1 \cup S_2 \cup S_3$. There is a vertex u in both segments $ve_1 \cdots e_2 v$ and $ve_3 \cdots e_0 v$ of E_1. Now the required cycle $C = F_1 F_2 F_3 F_4 F_5 F_6 F_1$ is as follows:

$$F_1 = e_0 ve_1 \cdots u \cdots e_2 ve_3 \cdots u \cdots e_0 = E_1,$$

$$F_2 = e_0 ve_2 \cdots u \cdots e_1 ve_3 \cdots u \cdots e_0 = E_2,$$

$$F_3 = e_0 ve_2 \cdots u \cdots e_3 ve_1 \cdots u \cdots e_0,$$

$$F_4 = e_0 ve_3 \cdots u \cdots e_2 ve_1 \cdots u \cdots e_0,$$

$$F_5 = e_0 ve_3 \cdots u \cdots e_1 ve_2 \cdots u \cdots e_0,$$

$$F_6 = e_0 ve_1 \cdots u \cdots e_3 ve_2 \cdots u \cdots e_0.$$

Case 2. $\deg v = 6$. Without loss of generality, we may assume that $E_1 = e_0 ve_1 \cdots e_2 ve_3 \cdots e_4 ve_5 \cdots e_0$ and $E_2 = e_0 ve_2 \cdots e_1 ve_3 \cdots e_4 ve_5 \cdots e_0$. Two subcases should be considered:

(1) $\{e_0, e_5\}$ is an edge cut of G. Then $V(\mathrm{Eu}(G)) = \bigcup_{j=1}^{4} S_j$. In this case, we may rewrite E_1 and E_2 as

$$\begin{aligned} E_1 &= e_2 ve_3 \cdots e_4 ve_5 \cdots e_0 ve_1 \cdots e_2 \\ &= e'_0 ve'_1 \cdots e'_2 ve'_3 \cdots e'_4 ve'_5 \cdots e'_0 \\ E_2 &= e_2 ve_0 \cdots e_5 ve_4 \cdots e_3 ve_1 \cdots e_2 \\ &= e'_0 ve'_4 \cdots e'_3 ve'_2 \cdots e'_1 ve'_5 \cdots e'_0. \end{aligned}$$

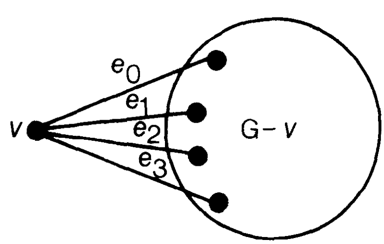

FIGURE 5

If $\{e'_0, e'_5\} = \{e_2, e_1\}$ is an edge cut of G, then $V(\mathrm{Eu}(G)) = \bigcup_{j=1}^{4} S_j$. The required cycle $C = F_1 F_2 \cdots F_8 F_1$ is defined as follows:

$$F_1 = e'_0 v e'_1 \cdots e'_2 v e'_3 \cdots e'_4 v e'_5 \cdots e'_0 = E_1$$
$$F_2 = e'_0 v e'_4 \cdots e'_3 v e'_2 \cdots e'_1 v e'_5 \cdots e'_0 = E_2$$
$$F_3 = e'_0 v e'_4 \cdots e'_3 v e'_1 \cdots e'_2 v e'_5 \cdots e'_0$$
$$F_4 = e'_0 v e'_2 \cdots e'_1 v e'_3 \cdots e'_4 v e'_5 \cdots e'_0$$
$$F_5 = e'_0 v e'_2 \cdots e'_1 v e'_4 \cdots e'_3 v e'_5 \cdots e'_0$$
$$F_6 = e'_0 v e'_3 \cdots e'_4 v e'_1 \cdots e'_2 v e'_5 \cdots e'_0$$
$$F_7 = e'_0 v e'_3 \cdots e'_4 v e'_2 \cdots e'_1 v e'_5 \cdots e'_0$$
$$F_8 = e'_0 v e'_1 \cdots e'_2 v e'_4 \cdots e'_3 v e'_5 \cdots e'_0.$$

If $\{e'_0, e'_5\} = \{e_2, e_1\}$ is not an edge cut of G, then there is a vertex u in G such that $E_1 = e_0 v e_1 \cdots u \cdots e_2 v e_3 \cdots u \cdots e_4 v e_5 \cdots e_0$. The required cycle $C = F_1 F_2 \cdots F_8 F_1$ is as follows:

$$F_1 = e_0 v e_1 \cdots u \cdots e_2 v e_3 \cdots u \cdots e_4 v e_5 \cdots e_0 = E_1$$
$$F_2 = e_0 v e_2 \cdots u \cdots e_1 v e_3 \cdots u \cdots e_4 v e_5 \cdots e_0 = E_2$$
$$F_3 = e_0 v e_2 \cdots u \cdots e_3 v e_1 \cdots u \cdots e_4 v e_5 \cdots e_0$$
$$F_4 = e_0 v e_3 \cdots u \cdots e_2 v e_1 \cdots u \cdots e_4 v e_5 \cdots e_0$$
$$F_5 = e_0 v e_3 \cdots u \cdots e_1 v e_2 \cdots u \cdots e_4 v e_5 \cdots e_0$$
$$F_6 = e_0 v e_4 \cdots u \cdots e_2 v e_1 \cdots u \cdots e_3 v e_5 \cdots e_0$$
$$F_7 = e_0 v e_4 \cdots u \cdots e_2 v e_3 \cdots u \cdots e_1 v e_5 \cdots e_0$$
$$F_8 = e_0 v e_1 \cdots u \cdots e_3 v e_2 \cdots u \cdots e_4 v e_5 \cdots e_0.$$

(2) $\{e_0, e_5\}$ is not an edge cut of G. Then $V(\mathrm{Eu}(G)) = \bigcup_{j=1}^{5} S_j$. In this case, there exists a vertex u in G such that $E_1 = e_0 v e_1 \cdots u \cdots e_4 v e_5 \cdots u \cdots e_0$. Now $C = F_1 F_2 \cdots F_{10} F_1$ is the required cycle, where

$$F_1 = e_0 v e_1 \cdots e_2 v e_3 \cdots e_4 v e_5 \cdots e_0 = E_1$$
$$F_2 = e_0 v e_2 \cdots e_1 v e_3 \cdots e_4 v e_5 \cdots e_0 = E_2$$
$$F_3 = e_0 v e_2 \cdots e_1 v e_4 \cdots e_3 v e_5 \cdots e_0$$
$$F_4 = e_0 v e_3 \cdots e_4 v e_1 \cdots e_2 v e_5 \cdots e_0$$
$$F_5 = e_0 v e_3 \cdots e_4 v e_2 \cdots e_1 v e_5 \cdots e_0$$

$$F_6 = e_0ve_4 \cdots e_3ve_2 \cdots e_1ve_5 \cdots e_0$$
$$= e_0ve_4 \cdots u \cdots e_1ve_5 \cdots u \cdots e_0$$
$$F_7 = e_0ve_4 \cdots u \cdots e_5ve_1 \cdots u \cdots e_0$$
$$F_8 = e_0ve_5 \cdots u \cdots e_4ve_1 \cdots u \cdots e_0$$
$$F_9 = e_0ve_5 \cdots u \cdots e_1ve_4 \cdots u \cdots e_0$$
$$F_{10} = e_0ve_1 \cdots u \cdots e_5ve_4 \cdots u \cdots e_0.$$

Case 3. $\deg v = 2t > 6$. First, we suppose $i = 2$, i.e., $E_1 = e_0ve_1 \cdots e_2ve_3 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$ and $E_2 = e_0ve_2 \cdots e_1ve_3 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$.

(1) $\{e_0, e_{2t-1}\}$ is an edge cut of G. Then $V(\mathrm{Eu}(G)) = \bigcup_{j=1}^{2(t-1)} S_j$. The required cycle $C = F_1F_2 \cdots F_{4n-1}F_{4n}F_{4n+1}F_{4n+2} \cdots F_{4t-6}F_{4t-5}F_{4t-4}F_1$ is defined as follows:

$$F_1 = e_0ve_1 \cdots e_2ve_3 \cdots e_{2t-2}ve_{2t-1} \cdots e_0 = E_1$$
$$F_2 = e_0ve_2 \cdots e_1ve_3 \cdots e_4ve_5 \cdots e_{2t-2}ve_{2t-1} \cdots e_0 = E_2$$
$$F_3 = e_0ve_2 \cdots e_1ve_4 \cdots e_3ve_5 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_4 = e_0ve_3 \cdots e_4ve_1 \cdots e_2ve_5 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_5 = e_0ve_3 \cdots e_4ve_2 \cdots e_1ve_5 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_6 = e_0ve_4 \cdots e_3ve_2 \cdots e_1ve_5 \cdots e_6ve_7 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_7 = e_0ve_4 \cdots e_1ve_6 \cdots e_5ve_7 \cdots e_{2t-2}ve_{2t-1} \cdots e_0$$
$$\vdots$$
$$F_{4n-1} = e_0ve_{2n} \cdots e_1ve_{2n+2} \cdots e_{2n+1}ve_{2n+3} \cdots \times e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_{4n} = e_0ve_{2n+1} \cdots e_{2n+2}ve_1 \cdots e_{2n}ve_{2n+3} \cdots \times e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_{4n+1} = e_0ve_{2n+1} \cdots e_{2n+2}ve_{2n} \cdots e_1ve_{2n+3} \cdots \times e_{2t-2}ve_{2t-1} \cdots e_0$$
$$F_{4n+2} = e_0ve_{2n+2} \cdots e_{2n+1}ve_{2n} \cdots e_1ve_{2n+3} \cdots \times e_{2t-2}ve_{2t-1} \cdots e_0$$
$$\vdots$$
$$F_{4t-7} = e_0ve_{2t-3} \cdots e_{2t-2}ve_{2t-4} \cdots e_1ve_{2t-1} \cdots e_0$$

$$F_{4t-6}=e_0ve_{2t-2}\cdots e_{2t-3}ve_{2t-4}\cdots e_3ve_2\cdots e_1ve_{2t-1}\cdots e_0$$

$$F_{4t-5}=e_0ve_{2t-2}\cdots e_{2t-3}ve_3\cdots e_{2t-4}ve_2\cdots e_1ve_{2t-1}\cdots e_0$$

$$F_{4t-4}=e_0ve_1\cdots e_2ve_{2t-4}\cdots e_3ve_{2t-3}\cdots e_{2t-2}ve_{2t-1}\cdots e_0.$$

(2) $\{e_0, e_{2t-1}\}$ is not an edge cut of G. Then $V(\mathrm{Eu}(G))=\bigcup_{j=1}^{2t-1} S_j$. Now there exists a vertex u with degree at least 4 in E_1 such that $E_1=e_0ve_1\cdots u\cdots e_{2t-2}ve_{2t-1}\cdots u\cdots e_0$. The required cycle $C=F_1F_2\cdots F_{4n-1}F_{4n}\cdots F_{4t-4}F_{4t-3}F_{4t-2}F_1$ is defined as follows:

Let $F_1, F_2,..., F_{4t-6}$ be the same as in (1). Now F_{4t-6} may be written as $e_0ve_{2t-2}\cdots u\cdots e_1ve_{2t-1}\cdots u\cdots e_0$. And we have

$$F_{4t-5}=e_0ve_{2t-2}\cdots u\cdots e_{2t-1}ve_1\cdots u\cdots e_0$$

$$F_{4t-4}=e_0ve_{2t-1}\cdots u\cdots e_{2t-2}ve_1\cdots u\cdots e_0$$

$$F_{4t-3}=e_0ve_{2t-1}\cdots u\cdots e_1ve_{2t-2}\cdots u\cdots e_0$$

$$F_{4t-2}=e_0ve_1\cdots u\cdots e_{2t-1}ve_{2t-2}\cdots u\cdots e_0.$$

If $i\neq 2$, let $i=2n+2$ for some n, $1\leqslant n\leqslant t-2$. Then

$$E_1=e_0ve_1\cdots e_2ve_3\cdots e_{2n+2}ve_{2n+3}\cdots e_{2t-2}ve_{2t-1}\cdots e_0$$

and

$$E_2=e_0ve_{2n+2}\cdots e_3ve_2\cdots e_1ve_{2n+3}\cdots e_{2t-2}ve_{2t-1}\cdots e_0.$$

In the above cycle C, we may relabel F_{4n-1} and F_{4n} as

$$\begin{aligned}F_{4n-1}&=e_0ve_{2n}\cdots e_1ve_{2n+2}\cdots e_{2n+1}ve_{2n+3}\cdots\\&\quad\times e_{2t-2}ve_{2t-1}\cdots e_0\\&=e'_0ve'_1\cdots e'_{2n}ve'_{2n+1}\cdots e'_{2n+2}ve'_{2n+3}\cdots\\&\quad\times e'_{2t-2}ve'_{2t-1}\cdots e'_0\end{aligned}$$

and

$$\begin{aligned}F_{4n}&=e_0ve_{2n+1}\cdots e_{2n+2}ve_1\cdots e_{2n}ve_{2n+3}\cdots\\&\quad\times e_{2t-2}ve_{2t-1}\cdots e_0\\&=e'_0ve'_{2n+2}\cdots e'_{2n+1}ve'_{2n}\cdots e'_1ve'_{2n+3}\cdots\\&\quad\times e'_{2t-2}ve'_{2t-1}\cdots e'_0\end{aligned}$$

It is obvious that $\{E_1, E_2\}$ and $\{F_{4n-1}, F_{4n}\}$ have the same form. It follows that starting with the Euler tour

$$F'_1=e_0ve_{2n}\cdots e_1ve_{2n+2}\cdots e_{2n+1}ve_{2n+3}\cdots e_{2t-2}ve_{2t-1}\cdots e_0$$

and repeating the above construction of cycle $C' = F'_1 F'_2 \cdots F'_{4t-1} \cdots F'_1$, we have $E_1 = F'_1 = F_{4n-1}$ and $E_2 = F'_2 = F_{4n}$. Thus the required cycle C' containing the edge $E_1 E_2$ is obtained. The proof is complete.

For a directed Eulerian graph a similar result can be obtained. Because of the concept of transformation in these two cases are quite different, we can not derive one result from other. We shall state the result of directed Euler tour graph in another paper.

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