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Z-TRANSFORMATION GRAPHS OF PERFECT MATCHINGS OF HEXAGONAL SYSTEMS

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Let H be a hexagonal system. We define the Z-transformation graph $Z(H)$ to be the graph where the vertices are the perfect matchings of H and where two perfect matchings are joined by an edge provided their symmetric difference is a hexagon of H. We prove that $Z(H)$ is a connected bipartite graph if H has at least one perfect matching. Furthermore, $Z(H)$ is either an elementary chain or graph with girth 4; and $Z(H) - V_m$ is 2-connected, where V_m is the set of monovalency vertices in $Z(H)$. Finally, we give those hexagonal systems whose Z-transformation graphs are not 2-connected.

In the past few years several kinds of transformation graphs were introduced such as tree graph [1], minimum tree graph [2], perfect matching polyhedra [3], matroid basis graph [4], Euler tour graph [5], and so on. In such a graph a vertex is a special sort of subgraph of a specified graph, and two vertices are joined by an edge provided that they can transform to each other by some specified transformation. Almost all of the above mentioned transformation graphs can be shown as polyhedra of (0, 1). Their properties have been extensively studied [6].

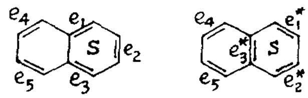
Recently some transformation graphs were studied and proved not to be polyhedra of (0, 1), such as generalized directed tree graphs [7] and hierarchical tree of Kekulé patterns of hexagonal systems [8].

In this paper we introduce a new kind of transformation graph, namely Z-transformation graph of perfect matchings of hexagonal systems and observe some of its properties. Note that this kind of transformation is not polyhedra of (0, 1).

A hexagonal system, also called "honeycomb system" or "hexanimal" (see e.g. [9]) is a finite connected plane graph with no cut-vertices in which every interior region is surrounded by a regular hexagon of side length 1 [10]. A perfect matching of a graph G is a set of disjoint edges of G covering all vertices of G.

In the present paper we confine oursleves to those hexagonal systems which have at least one perfect matching.

Definition 1. Let H be a hexagonal system with perfect matchings. The



$M_1=\{e_1, e_2, e_3, e_4, e_5\}$,

$M_2=\{e_1^*, e_2^*, e_3^*, e_4, e_5\}$,

$M_1 \Delta M_2 = s$.

Fig. 1.

Z-transformation graph $Z(H)$ is the graph where the vertices are the perfect matchings of H and where two perfect matchings M_1 and M_2 are joined by an edge provided that their symmetric difference $M_1 \triangle M_2$ is a hexagon of H. Let s be a hexagon of H and $M_1 \triangle M_2 = s$. Then $M_1 \triangle s = M_2$ and $M_2 \triangle s = M_1$. We call that M_1 and M_2 can be obtained from each other by a Z-transformation (see Fig. 1).

Let $s_1, s_2, \ldots, s_t$ be distinct hexagons of H. We use $H[s_1, s_2, \ldots, s_t]$ to denote the graph induced by the edges of $s_1, s_2, \ldots, s_t$.

Theorem 2. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is bipartite.*

Proof. If $Z(H)$ does not contain a cycle, then $Z(H)$ is bipartite.

Now assume that $Z(H)$ contains a cycle $C = M_0M_1 \ldots M_t$, where $M_0 = M_t$. Thus there exists a series of hexagons $s_1, \ldots, s_t$ such that $M_p = M_{p-1} \triangle s_p$ for $p = 1, \ldots, t$. Suppose that $s^1, \ldots, s^{m-1}$ and s^m are distinct hexagons among them, and s^i $(i = 1, \ldots, m)$ appears $\delta(s^i)$ times in $\{s_1, \ldots, s_t\}$. We now show that $\delta(s^i)$ is an even number for $i = 1, \ldots, m$. Note that if s^i contains an edge lying on the perimeter of H, then $\delta(s^i)$ must be even since $M_t = (\ldots((M_0 \triangle s_1) \triangle s_2)\ldots) \triangle s_t) = M_0$. Now suppose each edge of s^i does not lie on the perimeter of H. Since C is a cycle, there is no loss in generality in assuming that $s^i = s_1$. Let edge $e \in s_1$, and the other hexagon containing the edge e be s^*. Then we have $\delta(s^i) + \delta(s^*)$ is even. Assume that $\delta(s^i)$ is odd. Thereby $\delta(s^*)$ is odd too. Let $e^* \neq e$, $e^* \in s^*$, and e^* is parallel to e. Denote the other hexagon containing e^* by s^{**}. Then $\delta(s^{**})$ is odd too. Repeat this discussion, since H is finite, we eventually reach a hexagon $s' \in \{s^1, \ldots, s^m\}$ such that s' contains an edge lying on the perimeter of H and $\delta(s')$ is odd. This is contrary to the fact that for each hexagon s in $\{s^1, \ldots, s^m\}$ containing an edge lying on the perimeter of H, $\delta(s)$ must be even. This contradiction shows that $\delta(s^i)$ is even. Therefore $t = \sum_{i=1}^{m} \delta(s^i)$ is even, i.e. C is an even cycle.

The proof is completed. □

Theorem 3. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is connected.*

Proof. From the theorem in [8], we can deduce that any perfect matching of H is connected with the special perfect matching of H (called root Kekulé pattern), since every simultaneous rotation can be considered as a series of rotations of a single hexagon. So $Z(H)$ is connected.

In order to simplify the discussion, a hexagonal system is to be placed on a plane so that a pair of edges of each hexagon lie in parallel with the vertical line. Before continuing we review briefly some results about hexagonal systems. Let M be a perfect matching of a hexagonal system H. If six vertices of a hexagon s of H are covered by three edges of M, i.e. s is an M-alternating cycle, then s is called respectively, proper and improper sextet [12] (see Fig. 2). For each hexagonal system with perfect matchings, there exist exactly two perfect matchings, one of which has only proper sextets, the other of which has only improper sextets [12]. From this we can deduce that for each hexagonal system H, the Z-transformation graph $Z(H)$ has at most two vertices of monovalency. □

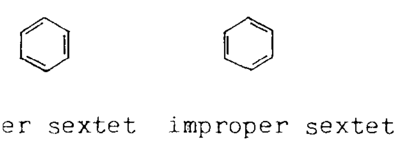


Fig. 2.

Let V_m be the set of monovalency vertices of $Z(H)$. Then we have the following.

Theorem 4. *Let H be a hexagonal system with perfect matchings. Then $Z(H)$ is either a path or a graph of girth 4 and $Z(H) - V_m$ is 2-connected.*

Proof. Suppose $Z(H)$ is not a path. Let $M \in V(Z(H)) - V_m$. We first show that H has at least two disjoint hexagons which are M-alternating cycles. If M has valency greater than two in $Z(H)$, then the above conclusion is evident. Now suppose that M has valency two in $Z(H)$, i.e. there are only two M-alternating cycles which are hexagons of H, say s_1 and s_2. If s_1 and s_2 are edge disjoint, then there is nothing to prove. The remainder case is that s_1 and s_2 have an edge in common (see Fig. 3). Let $M_1 = M \triangle s_2$. It is not difficult to verify that for M_1,

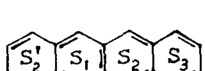


Fig. 3.

besides s_2, there is at most one hexagon, say s_3, which is an M_1-alternating cycle. In other words, M_1 has valency at most two in $Z(H)$. The same is true of $M_1' = M \triangle s_1$, i.e. there is at most one hexagon besides s_1, say s_2', which is an M_1'-alternating cycle. Note that $s_1 \cap s_2$, $s_2 \cap s_3$ and $s_1 \cap s_2'$ are in parallel with one another (see Fig. 3). Repeat this discussion, since H is finite, we can find a path $M_r'M_{r-1}' \ldots M_1'MM_1 \ldots M_{t-1}M_t$ ($r \geqslant 1$, $t \geqslant 1$) which is a component of $Z(H)$. By the connectivity of $Z(H)$, $Z(H)$ is a path itself and this contradicts the hypothesis. Therefore, M has at least two disjoint hexagons which are M-alternating cycles. This implies that if $Z(H)$ is not a path, for any $M \in Z(H) - V_m$ M is contained in a 4-cycle of $Z(H)$, namely $MM_1M_2M_3$, where $m_1 = M \triangle s_1$, $M_2 = M_1 \triangle s_2$ and $M_3 = M_2 \triangle s_1$, and s_1 and s_2 are edge disjoint hexagons of H which are M-alternating cycles. Since $Z(H)$ is bipartite, the girth of $Z(H)$ is 4.

In the following we shall prove that $Z(H) - V_m$ is 2-connected. Obviously, $Z(H) - V_m$ is connected. It suffices to prove the conclusion: for any 2-path of $Z(H)$, say $M_1M_2M_3$, there is another path $M_1M_2' \ldots M_3$ joining M_1 and M_3 which is internally vertex disjoint with $M_1M_2M_3$. In fact, if the above conclusion holds but $Z(H) - V_m$ is not 2-connected, then there is a cut vertex M of $Z(H) - V_m$. Take two vertices M' and M'' from different components of $Z(H) - V_m - M$ which are adjacent to M in $Z(H)$. Then there is no other path connecting M' and M'', a contradiction.

Now let $M_1M_2M_3$ be a 2-path in $Z(H)$, and $M_2 = M_1 \triangle s_1$, $M_3 = M_2 \triangle s_2$, where s_1 is a hexagon of H which is an M_1-alternating cycle, and s_2 is a hexagon of H which is an M_2-alternating cycle. If s_1 and s_2 are edge disjoint, then $M_1M_2'M_3$ is another path joining M_1 and M_3, where $M_2' = M_1 \triangle s_2$ and $M_3 = M_2' \triangle s_1$. If s_1 and s_2 are edge joint, then by previous discussion, there is another hexagon s_3 which is M_1-alternating cycle and is edge disjoint with s_1. If s_3 and s_2 are also edge disjoint, then there is another path $P' = M_1M_2'M_3'M_4'M_3$ joining M_1 and M_3, where $M_2' = M_1 \triangle s_3$, $M_3' = M_2' \triangle s_1$, $M_4' = M_3' \triangle s_2$ and $M_3 = M_4' \triangle s_3$. If s_3 and s_2 have an edge in common, since the degree of M_3 in $Z(H)$ is greater than 1, there is another hexagon s_4 which is an M_3-alternating cycle. It is not difficult to check that s_4 and s_2 are edge disjoint. If s_4 and s_1 are also edge disjoint, then we obtain a path as before joining M_1 and M_3. If s_4 and s_1 have an edge n in common (see Fig. 4), we can get another path $P'' = MM_1'M_2'M_3'M_4'M_5'M_3$, where $M_1' = M \triangle s_3$, $M_2' = M_1' \triangle s_1$, $M_3' = M_2' \triangle s_4$, $M_4' = M_3' \triangle s_3$, $M_5' = M_4' \triangle s_2$ and $M_3 = M_5' \triangle s_4$. The proof is complete. □

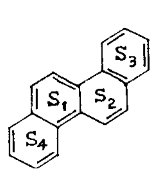


Fig. 4.

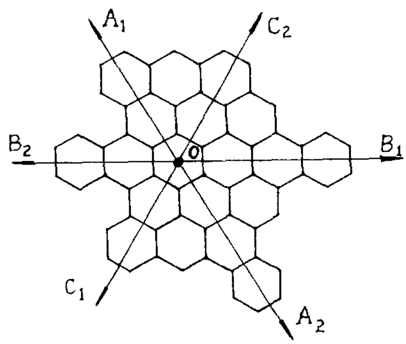


Fig. 5.

By the above theorem, a Z-transformation graph with $V_m = \emptyset$ is 2-connected. We now turn our attention to investigate when V_m is non-empty.

Let H be a hexagonal system, s_0 be a hexagon of H, O be the center of s_0. Draw three straight lines through O such that every line perpendicularly intersects a group of parallel edges of H. In fact, we obtain six half lines denoted as OA_1, OA_2, OB_1, OB_2, OC_1 and OC_2 (see Fig. 5). We call OA_i–OB_i–OC_i $(i = 1, 2)$ as coordinate system with respect to (briefly, w.r.t) s_0. Evidently, a coordinate system OA_i–OB_1–OC_i w.r.t. s_0 divides the plane into three areas A_iOB_i, B_iOC_i, C_iOA_i.

Let OA–OB–OC be a coordinate system w.r.t. s_0. For a point w lying in some area, say AOB, we define the coordinates of w to be the lengths of OW_A and OW_B (see Fig. 6), and denote $W(OA)$ and $W(OB)$, respectively.

A characteristic graph $T(H)$ of a hexagonal system H is defined to be the graph where vertex set is the set of the hexagons of H, and where two hexagons are joined by an edge provided they have a common edge. In fact, there is a natural way to draw the graph $T(H)$ of H. We can always let the vertices of $T(H)$ to be

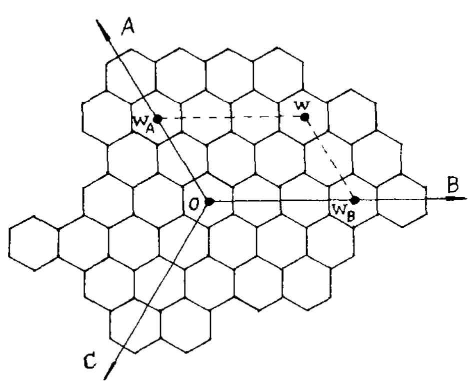
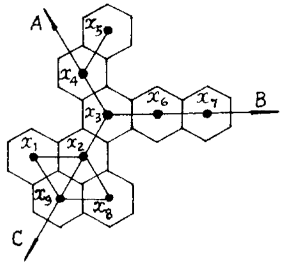


Fig. 6.



The boundary of T(H) is

$x_1x_2x_3x_4x_5x_4x_3x_6x_7x_6x_3x_2x_8x_9x_1$.

Fig. 7.

the centers of hexagons of H, two centers O_i and O_j are joined by edge provided the corresponding hexagons have a common edge (see Fig. 7).

Evidently, $T(H)$ is a planar graph. We define the perimeter of $T(H)$ to be the boundary of the exterior face, i.e. a close walk in which each cut edge of $T(H)$ is traversed twice (see Fig. 7).

Let $OA-OB-OC$ be a coordinate system of H. If the boundary of $T(H)$ lying in some area, say AOB, is a path $W_1W_2 \ldots W_t$ after deleting the edges lying in OA and OB and the path satisfies $W_1(OA) \geqslant W_2(OA) \geqslant \cdots \geqslant W_t(OA)$ and $W_1(OB) \leqslant W_2(OB) \leqslant \cdots \leqslant W_t(OB)$; or $W_1(OA) \leqslant W_2(OA) \leqslant \cdots \leqslant W_t(OA)$ and $W_1(OB) \geqslant W_2(OB) \geqslant \cdots \geqslant W_t(OB)$, then we call the perimeter of $T(H)$ to be monotone in area AOB. If the perimeter of $T(H)$ is monotone in all three areas, then $T(H)$ is called to be monotone w.r.t. the coordinate system $OA-OB-OC$ (see Fig. 8).

Let M be a perfect matching of H. M is called to be 3-dividable w.r.t. the coordinate system OA-OB-OC provided any edge of M does not intersect the lines OA, OB and OC, and two edges of M lie in the same area iff they are parallel (see Fig. 9).

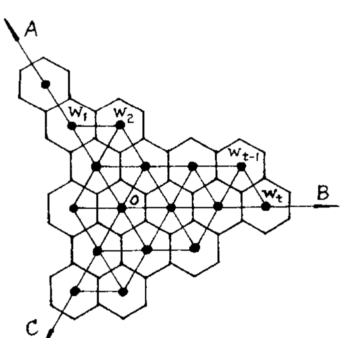


Fig. 8.

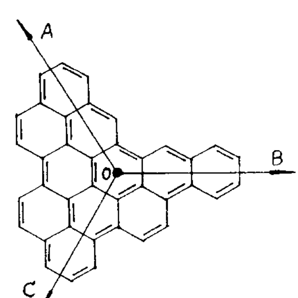


Fig. 9.

Lemma 5. *Let H be a hexagonal system, v be a vertex lying on the perimeter of H. If H or $H - v$ has a perfect matching M, then there is a hexagon which is an M-alternating cycle.*

Proof. by Lemma 2.1 in [11] or Lemma 2 in [13], the lemm is immediate. □

The following theorem will describe those hexagonal system whose Z-transformation graphs have one monovalecy vertex.

Theorem 6. *Let H be a hexagonal system. The following three statements are equivalent.*

(i) *$Z(H)$ has one monovalency vertex.*

(ii) *There exist a hexagon s_0 and a coordinate system OA–OB–OC w.r.t. s_0, and a perfect matching M of H such that M is 3-dividable w.r.t. the coordinate system OA-OB–OC.*

(iii) *There exist a hexagon s_0 and a coordinate system OA–OB–OC w.r.t. s_0 such that the perimeter of $T(H)$ is monotone w.r.t. the coordinate system OA–OB–OC.*

Proof. $(ii) \Rightarrow (i)$ is evident.

$(i) \Rightarrow (ii)$.

Let M be a monovalency vertex of $Z(H)$, s_0 be the only hexagon which is a M-alternating cycle. Let $OA–OB–OC$ be the coordinate system w.r.t. s_0 such that each of the three edges e_1, e_2 and e_3 of $M \cap s_0$ does not intersect the lines OA, OB and OC.

Suppose that M is not 3-dividable w.r.t. $OA–OB–OC$. Then in at least one area, say AOB, there is an edge $e \in M$ which is not parallel to e_1 (see Fig. 10). Since M is a monovalence vertex of $Z(H)$, it is not difficult to see that H has a series of hexagons $s_1, \ldots, s_t$ such that $e^* \in s_t$ lies on the perimeter of H, and the

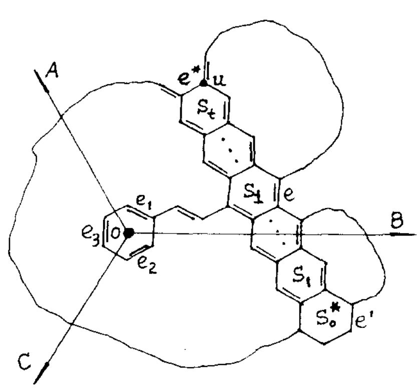


Fig. 10.

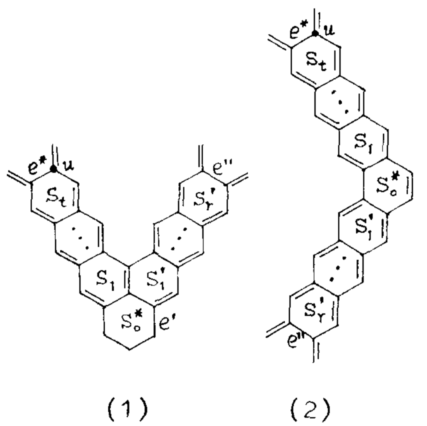


Fig. 11.

vertices of s_i ($1 \leqslant i \leqslant t$) are matched by the edges of M as depicted in Fig. 10. Let X be the set of all the hexagons of H. If the hexagon $s_0^* \notin H$, then the component H' of $H[X-\{s_1, \ldots, s_t\}]$ which contains the vertex u is a hexagonal system satisfying the conditions in Lemma 5. Hence there is another hexagon which is M-alternating cycle. This contradicts that M is *a* monovalency vertex in $Z(H)$.

If $s_0^* \in H$, according to $e' \notin M$ or $e' \in M$ (see Fig. 11), there are two possible cases, as shown in Fig. 11, where e'' lies on the perimeter of H. Let H' be the component of $H[X-\{s_1, \ldots, s_t, s_0^*, s_1', \ldots, s_r'\}]$ which contains the vetex u. By analogous arguments as above, both cases are impossible. Thus M is 3-dividable w.r.t. $OA-OB-OC$.

(ii) $\Rightarrow$ *(iii)*

By contradiction. Suppose that s_0, M and $OA-OB-OC$ mean just the same as in the statement (ii). But the perimeter of $T(H)$ is not monotone w.r.t. $OA-OB$-OC. Without loss of generality, we may assume in area AOB there are two vertices O_i and O_j of $T(H)$ lying on the perimeter of $T(H)$ with $O_i(OA) < O_j(OA)$ and $O_i(OB) < O_j(OB)$; moreover, edges e and e' are on the perimeter of H (see Fig. 12).

We can find a series of hexagons of H $s_j, s_1, \ldots, s_t$, ($t \geqslant 0$), such that the edge e'' is on the perimeter of H. Let the component of $H[X-\{s_j, s_1, \ldots, s_t\}]$ containing the hexagon s_j' be H_j. Evidently, H_j satisfies the conditions in Lemma 5 hence contains a hexagon which is a M-alternating cycle. This contradicts the statement (ii).

(iii) $\Rightarrow$ *(ii)*

By induction on the number of hexagons of H.

Suppose that there is a coordinate $OA-OB-OC$ w.r.t. a hexagon s_0 of H such that the perimeter of $T(H)$ is monotone w.r.t. the coordinate system $OA-OB-OC$. If H contains only one hexagon, there is nothing to prove. We now suppose that H contains more than one hexagon. If the vertices of $T(H)$ all lie on the axes

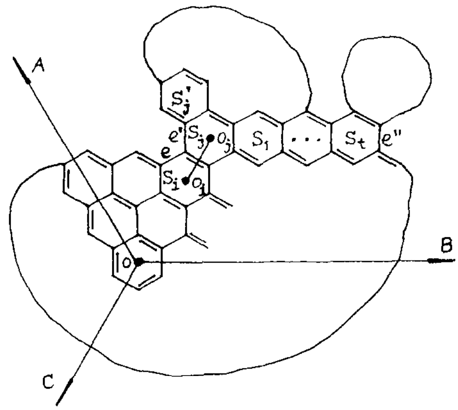


Fig. 12.

OA, OB and OC, then it is easy to check that the statement (ii) hold. We now assume that there is at least one vertex v on the perimeter of $T(H)$ not lying on the axes. Without loss of generality, we may assume v is in the area AOB (see Fig. 13), and $v'(OA) = v(OA)$, $v'(OB) < v(OB)$, $v''(OA) < v(OA)$, $v''(OB) = v(OB)$, where v' and v'' lie on the perimeter of $T(H)$ and are adjacent to v.

Let s_i be the hexagon of H whose center is v. Evidently the perimeter of $H[X - \{s_i\}]$ is still monotone w.r.t. the coordinate system OA–OB–OC. By the inductive hypothesis there exists a perfect matching M' which is 3-dividable w.r.t. OA–OB–OC. Therefore the perfect matcing $M' \cup e'$ of H is 3-dividable w.r.t. the coordinate system OA–OB–OC. This completes our proof. □

In the following we shall give a more intuitive description for a hexagonal system H whose Z-transformation graph has a monovalency vertex. Note that for any hexagonal system H there exists a smallest hexagon with its edges parallel to the edge of H and containing the interior of H in its interior. We denote it by $S(H)$.

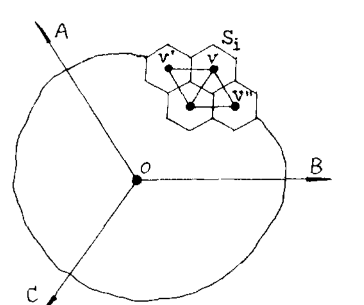


Fig. 13.

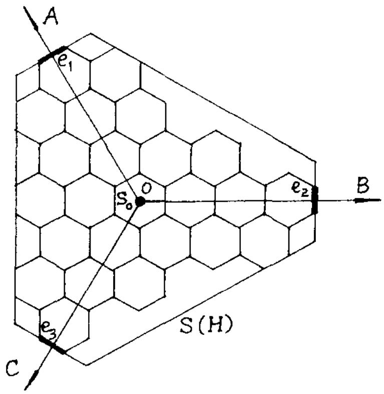


Fig. 14.

Theorem 7. *Let H be a hexagonal system. Then Z(H) contains a monovalency vertex iff the following three conditions are satisfied.*

(1) *There are three pairwise disjoint edges of S(H) each of which contains exactly an edge e_i of H ($i = 1, 2, 3$).*

(2) *The perpendicular bisectors of e_1, e_2 and e_3 intersect at the center 0 of some hexagon s_0 of H.*

(3) *The perimeter of T(H) is monotone w.r.t. the coordinate system OA–OB–OC shown in Fig.* 14.

Proof. By Theorem 6, it is easy to verify that if the conditions (1)–(3) are satisfied, then $Z(H)$ has a monovalency vertex.

Conversely, assume that $Z(H)$ has a monovalency vertex M. By Theorem 6, there exist a hexagon s_0 of H and a coordinate system $OA–OB–OC$ w.r.t. s_0 such that the perimeter of $T(H)$ is monotone w.r.t. $OA–OB–OC$. Moreover M is 3-dividable w.r.t. $OA–OB–OC$. It is not difficult to see that the edge e_i which is intersected by one of the coordinate axes OA, OB and OC and is the farthest from the center of s_0 must be contained in an edge L_i of $S(H)$ and L_i contains only one such edge of M. Evidently, the axis intersecting e_i is the perpendicular and bisector of e_i.

Now the proof is completed. □

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