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HAMILTON CYCLES IN DIRECTED EULER TOUR GRAPHS

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In this paper we define the directed Euler tour graph of a directed Eulerian graph by T-transformations, which was introduced by Xia Xin-guo in 1984, and prove that any edge in a directed Euler tour graph is contained in a Hamilton cycle.

Let $D=(V, X)$ be a directed Eulerian multigraph without loops, and let E be a directed Euler tour of D. For a vertex v of D with $\mathrm{id}(v)=\mathrm{od}(v)=k\geq 2$, E passes through v exactly k times. So we may write E: $x_0'vx_1\ldots x_1'vx_2\ldots x_2'v\ldots vx_k\ldots x_0'$, where $x_0', x_1', \ldots, x_{k-1}'$ are arcs going into v and $x_1, \ldots, x_k$ are arcs going out of v. A triple (x_{i-1}', v, x_i) is called a transition of E through v. A subsequence of E starting from v and ending at u (or v) which contains at least one arc is called a v-u (or v-v) segment of E. Let S and S' be two arc-disjoint v-u segments of E such that (S, S') is not a partition of E. We call S and S' to be exchangeable. A directed Euler tour F is said to be obtained from E by a T-transformation at S and S' if F is obtained from E by exchanging S and S'. The directed Euler tour graph of D, denoted by $\mathrm{Eu}(D)$, is an undirected simple graph defined as follows: The vertices of $\mathrm{Eu}(D)$ are directed Euler tours of D, and two directed Euler tours E and F are adjacent in $\mathrm{Eu}(D)$ if they can be obtained from each other by a T-transformation.

Xia Xin-guo [3] introduced the concept of the T-transformation of directed Euler tours and proved that any directed Euler tour graph is connected. In the present paper we prove that any directed Euler tour graph is edge-Hamiltonian as stated in the following.

Theorem. *Let D be a directed Euler graph having at least three directed Euler tours. Then any edge of* $\mathrm{Eu}(D)$ *is contained in a Hamilton cycle of* $\mathrm{Eu}(D)$.

Proof. For a cut vertex v of D with $\mathrm{id}(V)=2$ (see Fig. 1(a)), there are exactly two transitions (x_0', v, x_1) and (x_1', v, x_2) of E at v. Let D' be the graph obtained from D by replacing v by a pair of vertices v' and v'' (see Fig. 1(b)). It is easy to see that $\mathrm{Eu}(D)\cong\mathrm{Eu}(D')$. Hence we may assume that D has no cut vertex v with $\mathrm{id}(v)=2$.

Let Q be a subset of the vertex set of D such that $v\in Q$ if and only if $\mathrm{id}(v)\geq 2$.

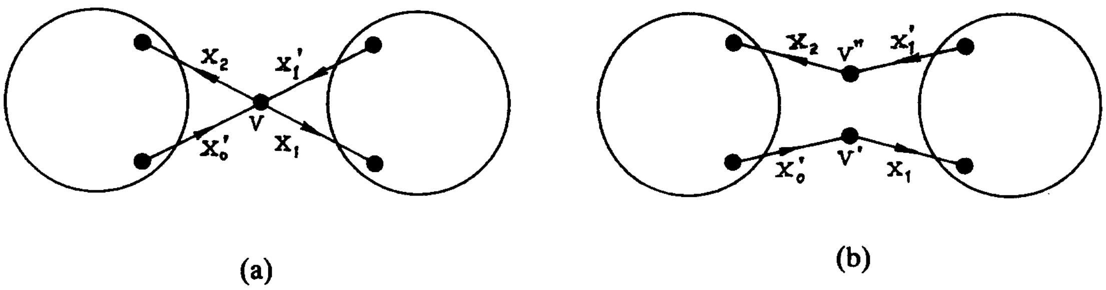

Fig. 1.

Let λ be the sum of indegrees of vertices in Q. The proof is by induction on λ. Since D has at least 3 Euler tours, we have $\lambda \geqslant 4$.

If $\lambda = 4$, then D is one of the graphs shown in Fig. 2.

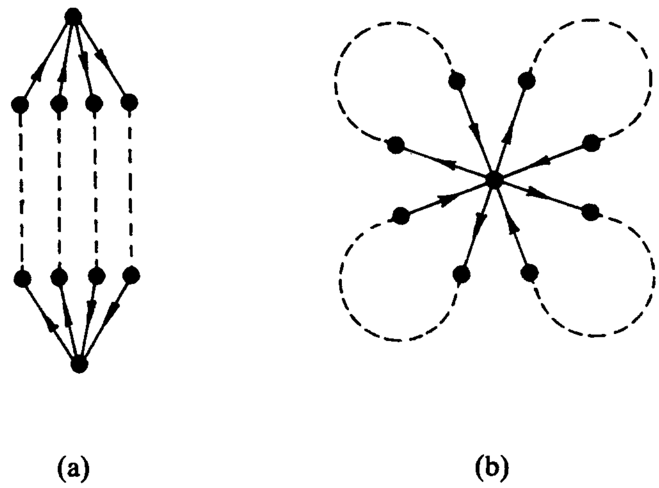

Fig. 2.

In case (a), $|V(\text{Eu}(D))| = 2$. In case (b), D has precisely 6 Euler tours, and it is easy to check that $\text{Eu}(D) = K_6$. The conclusion is evident.

Now suppose that the conclusion is true for $4 \leqslant \lambda \leqslant m$, where m is an integer. Let $\lambda = m + 1$. Take any edge E_1E_2 of $\text{Eu}(D)$, $E_1, E_2 \in V(\text{Eu}(D))$. By definition, E_2 is obtained from E_1 by a T-transformation, and vice versa. Two types of T-transformation are considered.

Type I. The T-transformation is carried out by exchanging two exchangeable v-v segments. We have

$$E_1 = x'_a v x_b \ldots x'_c v x_d \ldots x'_e v x_f \ldots x'_g v x_h \ldots x'_a$$

and

$$E_2 = x'_a v x_f \ldots x'_g v x_d \ldots x'_e v x_b \ldots x'_c v x_h \ldots x'_a.$$

We can relabel x'_a or x'_e as x'_0, and relabel the other arcs with v as its head or tail by $x_1, x'_1, x_2, x'_2, \ldots, x_k$ in accordance with the order arising in E_1. Because the T-transformation between E_1 and E_2 can be regarded as exchanging the positions of $vx_d \ldots x'_e v$ and $vx_h \ldots x'_a v$, we may also take x'_c or x'_g as x'_0. Then we rewrite E_1 and E_2 as follows.

$$E_1 = x'_0 \boldsymbol{vx_1 \ldots x'_l} vx_{l+1} \ldots x'_{i-1} \boldsymbol{vx_i \ldots x'_j} vx_{j+1} \ldots x'_{k-1} vx_k \ldots x'_0,$$

$$E_2 = x'_0 vx_i \ldots x'_j vx_{l+1} \ldots x'_{i-1} vx_1 \ldots x'_l vx_{j+1} \ldots x'_{k-1} vx_k \ldots x'_0,$$

where $1 \leqslant l < i \leqslant j \leqslant k-1$.

Type II. The T-transformation is carried out by exchanging two exchangeable v-u $(v \neq u)$ segments. As in Type I, we may label it as

$$E_1 = x'_0 \boldsymbol{vx_1 \ldots x'_{l-1} vx_l \ldots u} \ldots x'_l vx_{l+1} \ldots x'_{i-1} \boldsymbol{vx_i \ldots x'_{j-1} vx_j} \boldsymbol{\ldots u} \ldots x'_j vx_{j+1} \ldots x'_{k-1} vx_k \ldots x'_0,$$

$$E_2 = x'_0 vx_i \ldots x'_{j-1} vx_j \ldots u \ldots x'_l vx_{l+1} \ldots x'_{i-1} vx_1 \ldots x'_{l-1} vx_l \ldots u \ldots x'_j vx_{j+1} \ldots x'_{k-1} vx_k \ldots x'_0,$$

where $1 \leqslant l < i \leqslant j \leqslant k$. Because the T-transformation between E_1 and E_2 can be regarded as exchanging the positions of these two exchangeable u-v segments, we may also take the arc going into u as x'_0.

In both types we call v (or u) as a reference vertex and x'_0 as a reference arc.

Denote by S_i the set of directed Euler tour of D containing the transition (x'_0, v, x_i). Then it is obvious that $S_1, S_2, \ldots,$ form a partition of the vertex set of $\mathrm{Eu}(D)$. Let L_i be the subgraph of $\mathrm{Eu}(D)$ induced by S_i. Since L_i is isomorphic to the directed Euler tour graph of the directed graph which is obtained from D by replacing v by two vertices v' and v'' such that x'_0 and x_i are incident to v' and the other arcs incident to v in D are incident to v''. By the induction hypothesis, L_i is edge-Hamiltonian or isomorphic to K_1 (where $|S_i| = 1$) or K_2 (where $|S_i| = 2$).

Now we are going to find a cycle C in $\mathrm{Eu}(D)$ satisfying the following conditions.

(1) C contains E_1E_2;

(2) For each i, if $|S_i| > 1$, then C contains exactly one edge a_i in L_i, and if $|S_i| = 1$, then C contains exactly the vertex of S_i.

If there exists such a cycle C in $\mathrm{Eu}(D)$, we denote by H_i a Hamilton cycle containing the edge a_i in L_i (if $|S_i| \leqslant 2$, let $H_i = \emptyset$), then $(H_1 \cup H_2 \cup \cdots \cup H_j \cup \cdots) \Delta C$ is a Hamilton cycle containing edge E_1E_2 of $\mathrm{Eu}(D)$. Thereby, the theorem is proved. □

We consider the following three cases.

Case 1. $\mathrm{id}(v) = 2$.

E_2 can only be obtained from E_1 by exchanging two exchangeable v-u $(v \neq u)$ segments and $V(\mathrm{Eu}(D)) = S_1 \cup S_2$.

Subcase 1.1. $\mathrm{id}(u) \geqslant 3$

In this case, u occurs more than once in a v-v segment of E_1. We can choose a suitable reference arc such that

$$E_1 = x_0'vx_1 \ldots u \ldots u \ldots x_1'vx_2 \ldots u \ldots x_0'.$$

Then the required cycle $C = F_1F_2F_3F_4F_1$ is one of the following.

(1) $$F_1 = x_0'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots \mathbf{u} \ldots x_1'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u} \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_2 \ldots \mathbf{u} \ldots \mathbf{x}_1'\mathbf{v}\mathbf{x}_1 \ldots \underline{\mathbf{u} \ldots u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u} \ldots \mathbf{u} \ldots x_1'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots x_0',$$
$$F_4 = x_0'vx_1 \ldots \mathbf{u} \ldots \mathbf{x}_1'\mathbf{v}\mathbf{x}_2 \ldots \underline{\mathbf{u} \ldots u} \ldots x_0',$$

(2) $$F_1 = x_0'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots u \ldots x_1'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u} \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_2 \ldots \underline{u \ldots \mathbf{u}} \ldots \mathbf{x}_1'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u} \ldots x_1'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots u \ldots x_0',$$
$$F_4 = x_0'vx_1 \ldots \mathbf{u} \ldots \mathbf{x}_1'\mathbf{v}\mathbf{x}_2 \ldots \underline{\mathbf{u} \ldots u} \ldots x_0'.$$

Subcase 1.2. $\mathrm{id}(u) = 2$

Since D has at least three directed Euler tours, at least one of L_1 and L_2 has more than one vertex.

(1.2.1) If, say, $|S_1| = 1$, then $|S_2| \geqslant 2$

Let E_1 be the only directed Euler tour of S_1. Then for all u_i, $u_j \in Q - v$ there are no exchangeable u_i–u_j segments in E_1. Consequently, we have $\mathrm{id}(u_i) = 2$, $\mathrm{id}(u_j) = 2$, and there exists a vertex $u_1 \in Q - v - u$. We may choose a suitable reference arc such that $E_1 = x_0'vx_1 \ldots u \ldots u_1 \ldots x_1'vx_2 \ldots u_1 \ldots u \ldots x_0$. The required cycle $C = F_1F_2F_3F_1$ is as follows.

$$F_1 = x_0'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots \mathbf{u}_1 \ldots x'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u}_1 \ldots \mathbf{u} \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_2 \ldots \mathbf{u}_1 \ldots \mathbf{u} \ldots \mathbf{u}_1 \ldots \mathbf{x}_1'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\mathbf{v}\mathbf{x}_2 \ldots \mathbf{u}_1 \ldots x_1'\mathbf{v}\mathbf{x}_1 \ldots \mathbf{u} \ldots \mathbf{u}_1 \ldots u \ldots x_0'.$$

(1.2.2) Suppose that $|S_1| \geqslant 2$, $|S_2| \geqslant 2$

Then there exist two exchangeable u_{i1}-u_{j1} segments in E_1 and two exchangeable u_{i2}-u_{j2} segments in E_2, where u_{i1}, u_{i2}, u_{j1} and u_{j2} are in $Q - v - u$.

Let T_1, T_3 be the v-u segments in E_1; and T_2, T_4 be the u-v segments in E_1. If T_1 and T_3 (or T_2 and T_4) have an internal vertex $u_i \in Q - v - u$ in common, then the required cycle can be formed by exchanging v-u segments and u_i-u (or u-u_i) segments alternately. So we can assume that neither T_1 and T_3 nor T_2 and T_4 have an internal vertex in common. We now consider two cases.

(1.2.2.1) For u_i, $u_j \in Q - v - u$, there are two exchangeable u_i-u_j segments in E_1 (or E_2). We make the numbers of u_i's, u_j's in each of T_1, T_2, T_3 and T_4 as a quadruple (i_1, i_2, i_3, i_4), where $i_1+i_2+i_3+i_4=4$, which determines the distribution of u_i's and u_j's in E_1. Since any one of u and v can be taken as a reference vertex and any arc going into v or u can be taken as a reference arc, only one of the four quadruples (i_1, i_2, i_3, i_4), (i_2, i_3, i_4, i_1), (i_3, i_4, i_1, i_2) and (i_4, i_1, i_2, i_3) needs to be considered. Moreover, since we can take $F_1=E_2$ and $F_2=E_1$, only one of the two quadruples (i_1, i_2, i_3, i_4) and (i_3, i_2, i_1, i_4) needs to be considered. Therefore, we need to consider the following eight cases in total.

1. (1, 1, 1, 1), 2. (1, 1, 2, 0), 3. (1, 2, 1, 0), 4. (2, 2, 0, 0),
5. (2, 0, 2, 0), 6. (3, 1, 0, 0), 7. (3, 0, 1, 0), 8. (4, 0, 0, 0).

For cases 1, 2, 5, and 7 one can see that T_1 and T_3 or T_2 and T_4 have an internal vertex in common, which is contrary to our assumption. For Cases 4, 6, and 8, we shall form the cycle C from E_1 by exchanging v-u segments and u_i-u_j (u_j-u_i) segments alternately. For the Case 3, the required cycle C is as follows.

$$F_1=x_0'vx_1\ldots u_i\ldots u\ldots u_j\ldots u_i\ldots x_1'vx_2\ldots u_j\ldots u\ldots x_0'=E_1,$$
$$F_2=x_0'vx_2\ldots u_j\ldots u\ldots u_j\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u\ldots x_0'=E_2,$$
$$F_3=x_0'vx_2\ldots u_j\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u\ldots u_j\ldots u\ldots x_0',$$
$$F_4=x_0'vx_1\ldots u_i\ldots x_1'vx_2\ldots u_j\ldots u_i\ldots u\ldots u_j\ldots u\ldots x_0'.$$

(1.2.2.2) For any vertices $u_i, u_j \in Q-v-u$, there are no exchangeable u_i-u_j segments in both E_1 and E_2. Then $\mathrm{id}(u_i)=\mathrm{id}(u_j)=2$, and there are two exchangeable u-u_i (u_i-u) segments in E_1, and there are two exchangeable u-u_j (u_j-u) segments in E_2 at the same time.

Since neither T_1 and T_3 nor T_2 and T_4 have an internal vertex in common, we have

$$E_1=x_0'vx_1\ldots u_i\ldots u\ldots u_i\ldots x_1'vx_2\ldots u\ldots x_0',$$
$$E_2=x_0'vx_2\ldots u\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u\ldots x_0'.$$

Then u_j may appear in E_2 in the following manners.

(1) $E_2=x_0'vx_2\ldots u\ldots u_j\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u\ldots u_j\ldots x_0'$,

(2) $E_2=x_0'vx_2\ldots u\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u_j\ldots u\ldots u_j\ldots x_0'$,

(3) $E_2=x_0'vx_2\ldots u_j\ldots u\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u_j\ldots u\ldots x_0'$,

(4) $E_2=x_0'vx_2\ldots u_j\ldots u\ldots u_j\ldots u_i\ldots x_1'vx_1\ldots u_i\ldots u\ldots x_0'$.

It is not difficult to see that for each of Cases 1–4, there are exchangeable u_i-u_j (u_j-u_i) segments in E_1, contradicting the assumption of this subcase.

Case 2. $\mathrm{id}(v)=3$

Subcase 2.1. E_2 is obtained from E_1 by exchanging two exchangeable v-v

segments, i.e.,

$$E_1 = x_0'\mathbf{vx_1 \ldots x_1'}\underline{\mathbf{v}x_2 \ldots x_2'}vx_3 \ldots x_0',$$

$$E_2 = x_0'vx_2 \ldots x_2'vx_1 \ldots x_1'vx_3 \ldots x_0'.$$

(2.1.1) The vertex v is a cut vertex

We can take a suitable reference arc such that $\{x_0', x_3\}$ is an edge cut and $V(\mathrm{Eu}(D)) = S_1 \cup S_2$. Suppose $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ are edge cuts of D too. Note that $|V(\mathrm{Eu}(D))| \geqslant 3$. Then there is a v-v segment in which there exist two exchangeable u_i-u_j segments ($u_i, u_j \in Q - v - u$) and in which v only occurs as the end vertex of the v-v segment. Obviously, the required cycle $C = F_1 \ldots F_4F_1$ can be formed by exchanging v-v segments and u_i-u_j segments alternately. Now we suppose that $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ are not edge cuts of D. Then there is a vertex $u_1 \in Q - V$ arising in the segments $vx_1 \ldots x_1'v$ and $vx_2 \ldots x_2'v$. The required cycle $C = F_1 \ldots F_4F_1$ can be formed by exchanging v-v segments and $u_1 - v$ segments alternately.

(2.1.2) The vertex v is not a cut vertex and $V(\mathrm{Eu}(D)) = S_1 \cup S_2 \cup S_3$

Then there exists a vertex $u_1 \in Q$-v which arises in both the segments $vx_1 \ldots x_2'v$ and $vx_3 \ldots x_0'v$. If each v-v segment in E_1 contains the vertex u_1, then the required cycle $C = F_1F_2 \ldots F_6F_1$ is as follows.

$$F_1 = x_0'\mathbf{vx_1 \ldots u_1 \ldots x_1'v}x_2 \ldots \underline{u_1 \ldots x_2'}vx_3 \ldots u_1 \ldots x_0' = E_1,$$

$$F_2 = x_0'vx_2 \ldots \mathbf{u_1 \ldots x_2'vx_1 \ldots} \underline{\mathbf{u_1} \ldots x_1'vx_3 \ldots u_1} \ldots x_0' = E_2,$$

$$F_3 = x_0'\mathbf{vx_2 \ldots u_1 \ldots x_1'v}\underline{x_3 \ldots u_1 \ldots x_2'}vx_1 \ldots u_1 \ldots x_0',$$

$$F_4 = x_0'vx_3 \ldots \mathbf{u_1 \ldots x_2'vx_2 \ldots} \underline{\mathbf{u_1} \ldots x_1'vx_1 \ldots u_1} \ldots x_0',$$

$$F_5 = x_0'\mathbf{vx_3 \ldots u_1 \ldots x_1'v}\underline{x_1 \ldots u_1 \ldots x_2'}vx_2 \ldots u_1 \ldots x_0',$$

$$F_6 = x_0'vx_1 \ldots \mathbf{u_1 \ldots x_2'vx_3 \ldots} \underline{\mathbf{u_1} \ldots x_1'vx_2 \ldots u_1} \ldots x_0'.$$

If there is a v-v segment, say $vx_2 \ldots x_2'v$, which does not contain the vertex u_1, then there exists a vertex $u_2 \in Q - v - u_1$ which arises in both the segments $vx_2 \ldots x_2'v$ and $vx_3 \ldots x_1'v$. As before we consider the possible distribution of u_1's and u_2's in E_1. Note that the T-transformation between E_1 and E_2 can be regarded as exchanging any two exchangeable v-v segments in E_1, and we can put $F_1 = E_2$, $F_2 = E_1$. So we can choose a suitable reference arc such that E_1 and the required cycle $C = F_1F_2 \ldots F_6F_1$ are as follows.

$$F_1 = x_0'\mathbf{vx_1 \ldots u_2 \ldots u_1 \ldots x_1'v}\underline{x_2 \ldots u_2 \ldots x_2'}vx_3 \ldots u_1 \ldots x_0' = E_1,$$

$$F_2 = x_0'vx_2 \ldots \mathbf{u_2 \ldots x_2'vx_1} \ldots \mathbf{u_2 \ldots u_1 \ldots x_1'vx_3} \ldots u_1 \ldots x_0' = E_2,$$

$$F_3 = x_0'\mathbf{vx_2 \ldots u_2 \ldots u_1} \ldots x_1'vx_1 \ldots u_2 \ldots \mathbf{x_2'vx_3 \ldots u_1} \ldots x_0',$$

$$F_4 = x_0'vx_3 \ldots u_1 \ldots x_1'\mathbf{vx_1 \ldots u_2} \ldots x_2'\mathbf{vx_2 \ldots u_2} \ldots u_1 \ldots x_0',$$

$$F_5 = x_0'\mathbf{vx_3} \ldots \mathbf{u_1} \ldots \mathbf{x_1'vx_2} \ldots \mathbf{u_2} \ldots x_2'\mathbf{vx_1} \ldots \mathbf{u_2} \ldots u_1 \ldots x_0',$$
$$F_6 = x_0'vx_1 \ldots \mathbf{u_2} \ldots \mathbf{x_2'vx_3} \ldots \mathbf{u_1} \ldots x_1'vx_2 \ldots \mathbf{u_2} \ldots \mathbf{u_1} \ldots x_0'.$$

Subcase 2.2. E_2 is obtained from E_1 by exchanging two exchangeable v-u segments.

If $\mathrm{id}(u) = 2$, since we can take u as a reference vertex, it can be dealt with in the same way as in Case 1. If $\mathrm{id}(u) \geqslant 4$, it shall be considered later. Now we assume $\mathrm{id}(u) = 3$.

(2.2.1) The vertices u and v arise in E_1 alternately

By choosing a suitable reference vertex and a suitable reference arc, the required cycle $C = F_1F_2 \ldots F_6F_1$ is as follows.

$$F_1 = x_0'\mathbf{vx_1} \ldots \mathbf{u} \ldots x_1'\mathbf{vx_2} \ldots \mathbf{u} \ldots x_2'vx_3 \ldots u \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_2 \ldots \mathbf{u} \ldots \mathbf{x_1'vx_1} \ldots \underline{\mathbf{u} \ldots x_2'vx_3 \ldots u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\mathbf{vx_2} \ldots \mathbf{u} \ldots x_2'\mathbf{vx_3} \ldots \mathbf{u} \ldots x_1'vx_1 \ldots u \ldots x_0',$$
$$F_4 = x_0'vx_3 \ldots \mathbf{u} \ldots \mathbf{x_2'vx_2} \ldots \underline{\mathbf{u} \ldots x_1'vx_1 \ldots u} \ldots x_0',$$
$$F_5 = x_0'\mathbf{vx_3} \ldots \mathbf{u} \ldots x_1'\mathbf{vx_1} \ldots \mathbf{u} \ldots x_2'vx_2 \ldots u \ldots x_0',$$
$$F_6 = x_0'vx_1 \ldots u \ldots x_1'\mathbf{vx_3} \ldots \mathbf{u} \ldots x_2'\mathbf{vx_2} \ldots \mathbf{u} \ldots x_0'.$$

(2.2.2) Suppose that u does not arise in a v-v segment

We can choose a suitable reference vertex and a reference arc such that u does not arise in the segment $vx_2 \ldots x_2'v$ and $E_1 = x_0'vx_1 \ldots u \ldots u \ldots x_1'vx_2 \ldots x_2'vx_3 \ldots u \ldots x_0'$. Then the required cycle $C = F_1F_2 \ldots F_6F_1$ can be one of the following.

(1)
$$F_1 = x_0'\mathbf{vx_1} \ldots \mathbf{u} \ldots u \ldots x_1'vx_2 \ldots x_2'\mathbf{vx_3} \ldots \mathbf{u} \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_3 \ldots \underline{u \ldots \mathbf{u}} \ldots \mathbf{x_1'vx_2} \ldots \mathbf{x_2'vx_1} \ldots \mathbf{u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\mathbf{vx_3} \ldots \mathbf{u} \ldots \mathbf{x_1'}\underline{\mathbf{vx_2} \ldots x_2'vx_1} \ldots u \ldots u \ldots x_0',$$
$$F_4 = x_0'vx_2 \ldots x_2'vx_3 \ldots \mathbf{u} \ldots \mathbf{x_1'vx_1} \ldots \underline{\mathbf{u} \ldots u} \ldots x_0',$$
$$F_5 = x_0'\mathbf{vx_2} \ldots \mathbf{x_2'vx_3} \ldots \mathbf{u} \ldots \mathbf{u} \ldots x_1'\mathbf{vx_1} \ldots \mathbf{u} \ldots x_0',$$
$$F_6 = x_0'vx_1 \ldots \mathbf{u} \ldots \mathbf{x_1'vx_2} \ldots \mathbf{x_2'vx_3} \ldots \underline{\mathbf{u} \ldots u} \ldots x_0'.$$

(2)
$$F_1 = x_0'\mathbf{vx_1} \ldots \mathbf{u} \ldots u \ldots x_1'\mathbf{vx_2} \ldots \mathbf{x_2'vx_3} \ldots \mathbf{u} \ldots x_0' = E_1,$$
$$F_2 = x_0'vx_2 \ldots x_2'vx_3 \ldots \underline{u \ldots \mathbf{u}} \ldots \mathbf{x_1'vx_1} \ldots \mathbf{u} \ldots x_0' = E_2,$$
$$F_3 = x_0'\underline{vx_2 \ldots \mathbf{x_2'}}\mathbf{vx_3} \ldots \mathbf{u} \ldots \mathbf{x_1'vx_1} \ldots u \ldots u \ldots x_0',$$
$$F_4 = x_0'vx_3 \ldots \mathbf{u} \ldots \mathbf{x_1'vx_2} \ldots \mathbf{x_2'vx_1} \ldots \underline{\mathbf{u} \ldots u} \ldots x_0',$$
$$F_5 = x_0'\mathbf{vx_3} \ldots \mathbf{u} \ldots \mathbf{u} \ldots x_1'vx_2 \ldots x_2'\mathbf{vx_1} \ldots \mathbf{u} \ldots x_0',$$
$$F_6 = x_0'vx_1 \ldots \mathbf{u} \ldots \mathbf{x_1'vx_2} \ldots \mathbf{x_2'vx_3} \ldots \mathbf{u} \ldots u \ldots x_0'.$$

$$
\begin{aligned}
(3)\quad & F_1 = x'_0vx_1 \dots u \dots u \dots x'_1vx_2 \dots x'_2vx_3 \dots u \dots x'_0 = E_1,\\
& F_2 = x'_0vx_3 \dots u \dots x'_1vx_2 \dots x'_2vx_1 \dots \underline{u \dots u} \dots x'_0 = E_2,\\
& F_3 = x'_0vx_3 \dots u \dots u \dots x'_1\underline{vx_2 \dots x'_2}vx_1 \dots u \dots x'_0,\\
& F_4 = x'_0vx_2 \dots x'_2vx_3 \dots \underline{u \dots u} \dots x'_1vx_1 \dots u \dots x'_0,\\
& F_5 = x'_0vx_2 \dots x'_2vx_3 \dots u \dots x'_1vx_1 \dots u \dots u \dots x'_0,\\
& F_6 = x'_0vx_1 \dots u \dots x'_1vx_2 \dots x'_2cx_3 \dots \underline{u \dots u} \dots x'_0.
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad & F_1 = x'_0vx_1 \dots u \dots u \dots x'_1vx_2 \dots x'_2vx_3 \dots u \dots x'_0 = E_1,\\
& F_2 = x'_0vx_2 \dots x'_2vx_3 \dots u \dots x'_1vx_1 \dots \underline{u \dots u} \dots x'_0 = E_2,\\
& F_3 = x'_0\underline{vx_2 \dots x'_2}vx_3 \dots u \dots u \dots x'_1vx_1 \dots u \dots x'_0,\\
& F_4 = x'_0vx_3 \dots \underline{u \dots u} \dots x'_1vx_2 \dots x'_2vx_1 \dots u \dots x'_0,\\
& F_5 = x'_0vx_3 \dots u \dots x'_1vx_2 \dots x'_2vx_1 \dots u \dots u \dots x'_0.\\
& F_6 = x'_0vx_1 \dots u \dots x'_1vx_2 \dots x'_2vx_3 \dots \underline{u \dots u} \dots x'_0.
\end{aligned}
$$

Case 3. $\mathrm{id}(v) = k \geqslant 4$

Subcase 3.1. E_2 is obtained by exchanging to exchangeable v-v segments, i.e.,

$$
\begin{aligned}
E_1 &= x'_0vx_1 \dots x'_lvx_{l+1} \dots x'_{i-1}vx_i \dots x'_jvx_{j+1} \dots x'_{k-1}vx_k \dots x'_0 = F_1,\\
E_2 &= x'_0vx_i \dots x'_jvx_{l+1} \dots x'_{i-1}vx_1 \dots x'_lvx_{j+1} \dots x'_{k-1}vx_k \dots x'_0 = F_2,
\end{aligned}
$$

where $1 \leqslant l < i \leqslant j \leqslant k-1$.

(3.1.1) $\{x'_0, x_k\}$ is an edge cut of D, and $V(\mathrm{Eu}(D)) = \bigcup_1^{k-1} S_i$

The required cycle $C = F_1F_2 \dots F_{2k-2}F_1$ is as follows.

$$
\begin{aligned}
F_1 &= x'_0vx_1 \dots x'_lvx_{l+1} \dots x'_{i-1}vx_i \dots x'_jvx_{j+1} \dots x^r_{k-1}vx_k \dots x'_0 = E_1,\\
F_2 &= x'_0vx_i \dots x'_jvx_{l+1} \dots x'_{i-1}\underline{vx_1 \dots x^r_l}vx_{j+1} \dots x'_{k-1}vx_k \dots x'_0 = E_2,\\
F_3 &= x'_0vx_i \dots x'_jvx_1 \dots x'_1\underline{vx_2 \dots x'_2}vx_3 \dots x'_{i-1}vx_{j+1} \dots x'_{k-1}vx_k \dots x'_0,\\
F_4 &= x'_0vx_2 \dots x'_2\underline{vx_i \dots x'_j}vx_1 \dots x'_1vx_3 \dots x'_{i-1}vx_{j+1} \dots x'_{k-1}vx_k \dots x'_0,\\
F_5 &= x'_0vx_2 \dots x'_2vx_1 \dots x'_1\underline{vx_3 \dots x'_3}vx_4 \dots x'_{k-1}vx_k \dots x'_0,\\
F_6 &= x'_0vx_3 \dots x'_3vx_2 \dots x'_2\underline{vx_1 \dots x'_1}vx_4 \dots x'_{k-1}vx_k \dots x'_0,\\
F_7 &= x'_0vx_3 \dots x'_3vx_1 \dots x'_2\underline{vx_4 \dots x'_4}vx_5 \dots x'_{k-1}vx_k \dots x'_0,\\
F_8 &= x'_0vx_4 \dots x'_4vx_3 \dots x'_3\underline{vx_1 \dots x'_2}vx_5 \dots x'_{k-1}vx_k \dots x'_0,\\
&\vdots\\
F_{2i-1} &= x'_0vx_{i-1} \dots x'_{i-1}vx_1 \dots x'_{i-2}vx_i \dots x'_i\underline{vx_{i+1} \dots x'_{i+1}}vx_{i+2} \dots x'_{k-1}vx_k \dots x'_0,\\
F_{2i} &= x'_0vx_{i+1} \dots x'_{i+1}vx_{i-1} \dots x'_{i-1}\underline{vx_1 \dots x'_{i-2}vx_i \dots x'_i}vx_{i+2} \dots x'_{k-1}vx_k \dots x'_0,
\end{aligned}
$$

$$F_{2i+1}=x_0'vx_{i+1}\ldots x_{i+1}'vx_1\ldots x_i'\underline{vx_{i+2}\ldots x_{i+2}'}vx_{i+3}\ldots x_{k-1}'vx_k\ldots x_0',$$

$$\vdots$$

$$F_{2k-3}=x_0'vx_{k-1}\ldots x_{k-1}'\underline{vx_1\ldots x_1'}vx_2\ldots x_{k-2}'vx_k\ldots x_0',$$

$$F_{2k-2}=x_0'vx_1\ldots x_1'vx_{k-1}\ldots x_{k-1}'\underline{vx_2\ldots x_{k-2}'}vx_k\ldots x_0'.$$

(3.1.2) $\{x_0', x_k\}$ is not an edge cut of D, and $V(\mathrm{Eu}(D))=\bigcup_1^k S_i$

The sequence of F_i from F_1 to F_{2k-3} is the same as in (3.1.1). Because $\{x_0', x_k\}$ is not an edge cut of D, there is a vertex $u_1\in Q-v$ such that u_1 arises in both segments $vx_{k-1}\ldots x_{k-2}'v$ and $vx_k\ldots x_0'v$ in F_{2k-3}.

If u arises in the segment $vx_{k-1}\ldots x_{k-1}'v$, then we have

$$F_{2k-3}=x_0'vx_{k-1}\ldots u_1\ldots x_{k-1}'vx_1\ldots x_{k-2}'vx_k\ldots u_1\ldots x_0',$$

$$F_{2k-2}=x_0'vx_k\ldots u_1\ldots x_{k-1}'vx_1\ldots x_1'\underline{vx_2\ldots x_{k-2}'}vx_{k-1}\ldots u_1\ldots x_0',$$

$$F_{2k-1}=x_0'vx_k\ldots u_1\ldots x_{k-1}'vx_2\ldots x_{k-2}'vx_1\ldots x_1'vx_{k-1}\ldots u_1\ldots x_0',$$

$$F_{2k}=x_0'vx_1\ldots x_1'vx_{k-1}\ldots u_1\ldots x_{k-1}'\underline{vx_2\ldots x_{k-2}'}vx_k\ldots u_1\ldots x_0'.$$

If u_1 arises in the segment $vx_1\ldots x_1'v$ or $vx_2\ldots x_{k-2}'v$, we can obtain the required cycle $C=F_1F_2\ldots F_{2k}F_1$ in a similar way as above.

Note that if $i=k-1$, then

$$F_{2k-3}=x_0'vx_{k-2}\ldots x_{k-2}'vx_1\ldots x_{k-3}'vx_{k-1}\ldots x_{k-1}'vx_k\ldots x_0'.$$

If u_1 arises in the segment $x_{k-2}\ldots x_{k-2}'$, then

$$F_{2k-3}=x_0'vx_{k-2}\ldots u_1\ldots x_{k-2}'vx_1\ldots x_{k-3}'vx_{k-1}\ldots x_{k-1}'vx_k\ldots u_1\ldots x_0',$$

$$F_{2k-2}=x_0'vx_k\ldots u_1\ldots x_{k-2}'vx_1\ldots x_{k-3}'\underline{vx_{k-1}\ldots x_{k-1}'}vx_{k-2}\ldots u_1\ldots x_0',$$

$$F_{2k-1}=x_0'vx_k\ldots u_1\ldots x_{k-2}'vx_{k-1}\ldots x_{k-1}'vx_1\ldots x_{k-3}'vx_{k-2}\ldots u_1\ldots x_0',$$

$$F_{2k}=x_0'vx_1\ldots x_{k-3}'vx_{k-1}\ldots x_{k-1}'vx_k\ldots u_1\ldots x_{k-2}'vx_{k-2}\ldots u_1\ldots x_0'.$$

If u_1 arises in the segment $vx_1\ldots x_{k-3}'v$ or $vx_{k-1}\ldots x_{k-1}'v$, we can obtain the required cycle $C=F_1F_2\ldots F_{2k}F_1$ in a similar way as above.

Subcase 3.2. E_2 is obtained by exchanging two exchangeable v-u segments, i.e.,

$$E_1=x_0'vx_1\ldots x_{l-1}'vx_l\ldots u\ldots x_l'vx_{l+1}\ldots x_{i-1}'vx_i\ldots x_{j-1}'vx_j\ldots u\ldots x_j'vx_{j+1}\ldots x_{k-1}'vx_k\ldots x_0'=F_1,$$

$$E_2=x_0'vx_i\ldots x_{j-1}'vx_j\ldots u\ldots x_l'vx_{l+1}\ldots x_{i-1}'vx_1\ldots x_{l-1}'vx_l\ldots u\ldots x_j'vx_{j+1}\ldots x_{k-1}'vx_k\ldots x_0'=F_2,$$

where $1\leqslant l<i\leqslant j\leqslant k$.

(3.2.1) $\{x_0', x_k\}$ is an edge cut, and $V(\mathrm{Eu}(D)) = \bigcup_1^{k-1} S_i$

In a similar way as in Subcase (3.1.1), we can form the sequence $F_2, F_3, \ldots, F_{2k-2}$ from F_2 such that

$$F_{2k-2} = x_0'vx_1 \ldots x_1'\boldsymbol{vx_i} \ldots \boldsymbol{x_{j-1}'vx_j} \ldots \boldsymbol{u} \ldots x_l'vx_{l+1} \ldots$$
$$x_{i-1}'\boldsymbol{vx_2} \ldots \boldsymbol{x_{l-1}'vx_l} \ldots \boldsymbol{u} \ldots x_j'vx_{j+1} \ldots x_{k-1}'vx_k \ldots x_0'.$$

(3.3.2) $\{x_0', x_k\}$ is not an edge cut, and $V(\mathrm{Eu}(D)) = \bigcup_1^k S_i$

From F_2 we form the sequence $F_2, F_3, \ldots, F_{2k-4}$ such that

$$F_{2k-4} = x_0'vx_{k-1} \ldots x_{k-1}'\boldsymbol{vx_i} \ldots \boldsymbol{x_{j-1}'vx_j} \ldots \boldsymbol{u} \ldots x_l'vx_{l+1} \ldots$$
$$x_{i-1}'\boldsymbol{vx_1} \ldots \boldsymbol{x_{l-1}'vx_l} \ldots \boldsymbol{u} \ldots x_j'vx_{j+1} \ldots x_{k-2}'vx_k \ldots x_0',$$
$$F_{2k-3} = x_0'vx_{k-1} \ldots x_{k-1}'vx_1 \ldots x_{k-2}'vx_k \ldots x_0'.$$

Because $\{x_0', x_k\}$ is not an edge cut of D, then there exists a vertex $u_1 \in Q - v$ such that

$$F_{2k-3} = x_0'vx_{k-1} \ldots u_1 \ldots x_{k-2}'vx_k \ldots u_1 \ldots x_0'.$$

Furthermore, the sequence $F_{2k-3}, F_{2k-2}, \ldots, F_{2k}, F_1$ is the same as in Subcase (3.1.2).

The proof is complete. □

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