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3D point force solution for a permeable penny-shaped crack embedded in an infinite transversely isotropic piezoelectric medium

W.Q. CHEN[1,2] and C.W. LIM[3,*]
[1]*Department of Civil Engineering, Zhejiang University, Hangzhou 310027, P.R. China*
[2]*State Key Lab of CAD and CG, Zhejiang University, Hangzhou 310027, P.R. China*
[3]*Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, P.R. China*
[*]*Author for correspondence (E-mail: bccwlim@cityu.edu.hk)*

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Abstract. This paper considers the non-axisymmetric three-dimensional problem of a penny-shaped crack with permeable electric conditions imposed on the crack surfaces, subjected to a pair of point normal forces applied symmetrically with respect to the crack plane. The crack is embedded in an infinite transversely isotropic piezoelectric body with the crack face perpendicular to the axis of material symmetry. Applying the symmetry of the problem under consideration then leads to a mixed–mixed boundary value problem of a half-space, for which potential theory method is employed for the purpose of analysis. The cases of equal eigenvalues are also discussed. Although the treatment differs from that for an impermeable crack reported in literature, the resulting governing equation still has a familiar structure. For the case of a point force, exact expressions for the full-space electro-elastic field are derived in terms of elementary functions with explicit stress and electric displacement intensity factors presented. The exact solution for a uniform loading is also given.

Key words: Exact point force solution, penny-shaped crack, permeable crack, piezoelectric medium, potential theory.

1. Introduction

The fracture of piezoelectric materials has gained intense research interest in recent decades due to the susceptibility of piezoceramics, the most commercial piezoelectric materials of practical importance, to cracking. The review papers of Zhang et al. (2002), Chen and Lu (2003) and Zhang and Gao (2004) have given an excellent description of the state-of-the-art of fracture mechanics of piezoelectric materials in various aspects including thermodynamic approaches, two-dimensional cracks, interface cracks, three-dimensional cracks, fracture criteria, electric conditions along the crack face, and experimental observations, etc. The readers are referred to the above three papers for a long list of relevant references. Those mentioned here are the electric conditions at crack surfaces adopted in the literature. According to Chen and Lu (2003), there are currently five types of boundary conditions at the crack surfaces, i.e. (a) impermeable cracks with traction-free surfaces; (b) permeable cracks with traction-free surfaces; (c) cracks with exact electric boundary and traction-free surfaces; (d) impermeable cracks with near-tip microstructural features; and (e) impermeable (or permeable) cracks with contacting surfaces. The last two types of conditions are generally non-linear and are introduced to study more complicated effects such as electric-field induced yielding and crack closure. The first three types of conditions are linear, and the impermeable crack and the permeable crack are actually two limiting cases of the crack with exact electric boundary conditions (Zhang and Tong, 1996). Although it is more reasonable to adopt the exact electric boundary conditions in linear fracture analysis of piezoelectric materials, as evidenced by the recent experimental observation (Schneider et al., 2003), the corresponding theoretical analysis is very complicated, and it is very difficult to obtain an exact solution for a general three-dimensional crack. On the other hand, the upper and lower bounds of some fracture parameters, for example the crack opening displacement, of piezoelectric materials can be determined when the impermeable and permeable crack solutions are known. Thus, it is still of theoretical importance to acquire analytical solutions of both impermeable and permeable cracks.

The method of potential theory proposed by Fabrikant (1986) has been applied extensively in engineering and many new exact three-dimensional solutions of mixed and mixed–mixed boundary value problems for transversely isotropic elasticity have been reported (Fabrikant, 1989, 1991). In recent years, Chen et al. (Chen and Shioya, 1999a,b, 2000; Chen et al., 2000, 2001a,b) and Hou et al. (2001) applied Fabrikant's method to piezoelasticity and obtained several important exact three-dimensional solutions of crack problems. The key point for successful application of Fabrikant's method is to introduce an additional potential of a simple layer to account for the electric field in piezoelectric materials (Chen and Ding, 2004). It was found that the resulting governing integro-differential equations have the same structure as that appearing in the elastic fracture analysis of transversely isotropic materials. Thus, the new results in potential theory derived by Fabrikant (1989, 1991) can be utilized directly. However, the analyses mentioned above were all confined to impermeable cracks.

Although there are a lot of two-dimensional studies on permeable cracks (Zhang and Gao, 2004), the corresponding three-dimensional work is still very limited. Yang and Lee (2001) considered the problem of a permeable penny-shaped crack in a piezoelectric strip. Later, they studied the problem of a permeable penny-shaped crack in a piezoelectric cylinder surrounded by an infinite elastic medium (Yang and Lee, 2003). Recently, Li and Lee (2004) presented an exact solution for an external circular crack with permeable electric conditions. It is noted that, although different problems were solved, the above three papers are all limited to the axisymmetric case and hence the Hankel transform technique was widely used. There is an alternative way to derive the solution of a permeable penny-shaped crack in an infinite piezoelectric material subjected to remote uniform loading (Kogan et al., 1996). This method can be named as spheroidal-inclusion-approach since the solution of the penny-shaped crack is obtained from that of a spheroidal inclusion through a limiting procedure (there is a corresponding term "elliptical-cavity-approach" in the two-dimensional case according to Zhang and Gao (2004)). The only investigation on non-axisymmetric permeable crack was conducted by Yang and Lee (2002), who employed Hankel transform as well as Fourier series expansion to derive an analytical solution in the transformed domain for a penny-shaped crack in a piezoelectric strip. Numerical methods were employed for the inversion of transformed field variables. Hitherto, there are no exact solutions available in literature for non-axisymmetric problems (mode I) of permeable cracks. It is also noted that the results based on permeable and impermeable assumptions are identical for mode II and mode III problems of a penny-shaped crack as shown in Chen and Shioya (2000).

The non-axisymmetric problem (mode I) of a permeable penny-shaped crack embedded in an infinite transversely isotropic piezoelectric medium is studied in this paper by virtue of the method of potential theory proposed by Fabrikant (1989). In contrast to the analysis for impermeable cracks (Chen and Shioya, 1999a), the harmonic functions in the general solution are expressed in terms of a potential of a simple layer only. For an arbitrarily shaped flat crack subjected to arbitrarily distributed symmetric normal forces, we arrive at an integro-differential equation, of which the structure is exactly the same as that for an impermeable crack or an elastic crack. The point force solution is then obtained using the available results in potential theory and all expressions for the full-space electro-elastic field are expressed in terms of elementary functions. The exact solution for a uniformly distributed load is also presented. The analysis procedure for equal eigenvalues is also outlined and discussed.

2. The General solution of transversely isotropic piezoelasticity

The Cartesian coordinates (x, y, z) is selected such that the z-axis is parallel to the axis of symmetry of a transversely isotropic piezoelectric body. As shown in Ding et al. (1996, 1997), the form of general solution depends greatly on the relationships among the three characteristic roots with positive real part, denoted as $s_i (i=1,2,3)$, of the following algebraic equation

$$n_0 s^6 - n_1 s^4 + n_2 s^2 - n_3 = 0, \tag{1}$$

where the coefficients $n_i (i=0,1,2,3)$ are related to the elastic constants c_{ij}, piezoelectric constants e_{ij}, and dielectric constants ε_{ij} (Chen and Shioya, 1999a).

2.1. Case for distinct s_i

We denote the three components of displacement as u, v and $w \equiv w_1$ in x-, y- and z-directions, respectively, and the electric potential as $\Phi \equiv w_2$. By introducing a complex tangential displacement field $U = u + \mathrm{i}v$, the general solution reads as (Ding et al., 1997; Ding and Chen, 2001)

$$U = \Lambda \left(\sum_{i=1}^{3} F_i + \mathrm{i} F_4 \right), \quad w_k = \sum_{i=1}^{3} \alpha_{ik} \frac{\partial F_i}{\partial z_i} \quad (k=1,2), \tag{2}$$

where $\Lambda = \partial/\partial x + \mathrm{i}\partial/\partial y$ and $\mathrm{i} = \sqrt{-1}$. Here and subsequently, α_{ik} are constants defined in Ding et al. (1997) which are also available in Chen and Shioya (1999a), and $F_i (i=1,2,3,4)$ are quasi-harmonic functions satisfying

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2} \right) F_i = 0, \quad (i=1,2,3,4), \tag{3}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $z_i = s_i z$ with $s_4 = \sqrt{c_{66}/c_{44}}$ being another material eigenvalue.

The expressions for stresses and electric displacements are easily obtained from the constitutive relations (Ding and Chen, 2001):

$$\sigma_{zj} = \sum_{i=1}^{3} \gamma_{ij} \frac{\partial^2 F_i}{\partial z_i^2} \quad (j=1,2,3),$$
$$\sigma_2 = 2c_{66}\Lambda^2 \left(\sum_{i=1}^{3} F_i + \mathrm{i}F_4 \right),$$
$$\tau_{zk} = \Lambda \left(\sum_{i=1}^{3} \gamma_{ik} s_i \frac{\partial F_i}{\partial z_i} + \mathrm{i}\vartheta_k \frac{\partial F_4}{\partial z_4} \right) \quad (k=1,2), \tag{4}$$

where, $\sigma_{z1} = \sigma_z$, $\sigma_{z2} = D_z$, $\sigma_{z3} = \sigma_x + \sigma_y$, $\sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy}$, $\tau_{z1} = \tau_{xz} + i\tau_{yz}$, $\tau_{z2} = D_x + \mathrm{i}D_y$, and

$$\gamma_{i1} = -c_{13} + c_{33} s_i \alpha_{i1} + e_{33} s_i \alpha_{i2}, \quad \gamma_{i2} = -e_{31} + e_{33} s_i \alpha_{i1} - \varepsilon_{33} s_i \alpha_{i2},$$
$$\gamma_{i3} = 2[(c_{66} - c_{11}) + c_{13} s_i \alpha_{i1} + e_{31} s_i \alpha_{i2}], \quad \vartheta_1 = s_4 c_{44}, \quad \vartheta_2 = s_4 e_{15} \tag{5}$$

Throughout this paper, the subscript j always ranges from 1 to 3, while the subscript k is taken to be 1 and 2 only. This rule will not be indicated hereafter.

2.2. Case for $s_1 \neq s_2 = s_3$

The general solution is (Ding et al., 1997; Ding and Chen, 2001)

$$U = \Lambda \left(\sum_{i=1}^{2} F_i + z_2 F_3 + \mathrm{i}F_4 \right),$$
$$w_k = \sum_{i=1}^{2} \alpha_{ik} \frac{\partial F_i}{\partial z_i} + \alpha_{2k} z_2 \frac{\partial F_3}{\partial z_2} + \alpha_{4k} F_3. \tag{6}$$

The expressions for stresses and electric displacements are obtained as

$$\sigma_{zj} = \sum_{i=1}^{2} \gamma_{ij} \frac{\partial^2 F_i}{\partial z_i^2} + \gamma_{2j} z_2 \frac{\partial^2 F_3}{\partial z_2^2} + \gamma_{4j} \frac{\partial F_3}{\partial z_2},$$
$$\sigma_2 = 2c_{66}\Lambda^2 \left(\sum_{i=1}^{2} F_i + z_2 F_3 + \mathrm{i}F_4 \right),$$
$$\tau_{zk} = \Lambda \left(\sum_{i=1}^{2} \gamma_{ik} s_i \frac{\partial F_i}{\partial z_i} + \gamma_{2k} s_2 z_2 \frac{\partial F_3}{\partial z_2} + \omega_{1k} F_3 + \mathrm{i}\vartheta_k \frac{\partial F_4}{\partial z_4} \right), \tag{7}$$

where

$$\gamma_{41} = s_2[c_{33}(\alpha_{21} + \alpha_{41}) + e_{33}(\alpha_{22} + \alpha_{42})], \quad \gamma_{42} = s_2[e_{33}(\alpha_{21} + \alpha_{41}) - \varepsilon_{33}(\alpha_{22} + \alpha_{42})],$$
$$\gamma_{43} = 2s_2[c_{13}(\alpha_{21} + \alpha_{41}) + e_{31}(\alpha_{22} + \alpha_{42})],$$
$$\omega_{11} = c_{44}(s_2 + \alpha_{41}) + e_{15}\alpha_{42}, \quad \omega_{12} = e_{15}(s_2 + \alpha_{41}) - \varepsilon_{11}\alpha_{42}. \tag{8}$$

2.3. Case for $s_1 = s_2 = s_3$

The general solution is (Ding et al., 1997; Ding and Chen, 2001)

$$U = \Lambda\left(F_1 + z_1 F_2 + z_1^2 \frac{\partial F_3}{\partial z_1} + \mathrm{i}F_4\right),$$
$$w_k = \alpha_{1k}\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \alpha_{4k}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \alpha_{5k}F_3. \tag{9}$$

The expressions for stresses and electric displacements are obtained as

$$\sigma_{zj} = \gamma_{1j}\left(\frac{\partial^2 F_1}{\partial z_1^2} + z_1\frac{\partial^2 F_2}{\partial z_1^2} + z_1^2\frac{\partial^3 F_3}{\partial z_1^3}\right) + \gamma_{4j}\left(\frac{\partial F_2}{\partial z_1} + 2z_1\frac{\partial^2 F_3}{\partial z_1^2}\right) + \gamma_{5j}\frac{\partial F_3}{\partial z_1},$$
$$\sigma_2 = 2c_{66}\Lambda^2\left(F_1 + z_1 F_2 + z_1^2\frac{\partial F_3}{\partial z_1} + \mathrm{i}F_4\right), \tag{10}$$
$$\tau_{zk} = \Lambda\left[\gamma_{1k}s_1\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \omega_{1k}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \omega_{2k}F_3 + \mathrm{i}\vartheta_k\frac{\partial F_4}{\partial z_4}\right],$$

where

$$\begin{aligned}
\gamma_{51} &= s_1[c_{33}(2\alpha_{41} + \alpha_{51}) + e_{33}(2\alpha_{42} + \alpha_{52})],\\
\gamma_{52} &= s_2[e_{33}(2\alpha_{41} + \alpha_{51}) - \varepsilon_{33}(2\alpha_{42} + \alpha_{52})],\\
\gamma_{53} &= 2s_1[c_{13}(2\alpha_{41} + \alpha_{51}) + e_{31}(2\alpha_{42} + \alpha_{52})],\\
\omega_{21} &= c_{44}\alpha_{51} + e_{15}\alpha_{52}, \quad \omega_{22} = e_{15}\alpha_{51} - \varepsilon_{11}\alpha_{52}.
\end{aligned} \tag{11}$$

3. The Method of Potential Theory for Permeable Crack

Consider a flat crack of arbitrary shape S located in the plane $z = 0$ in a transversely isotropic piezoelectric medium. It is assumed that two equal and opposite arbitrary normal pressures p act symmetrically to the upper and lower crack faces. Note that, external electromechanical loading is generally not applied to the crack faces directly. However, by using the principle of superposition, the total solution can be obtained by combining the following solution and the trivial solution of an infinite piezoelectric medium subjected to the applied loads without disturbance of the crack. Note that the latter solution does not contribute to the singular stress field and/or electric displacement field due to the crack, and thus only the former solution is discussed below.

Contrary to the impermeable assumption adopted in Chen and Shioya (1999a), here we deal with the permeable conditions, i.e. the normal electric displacement and the electric potential are both continuous across the crack:

$$D_z(x, y, 0^+) = D_z(x, y, 0^-), \quad \Phi(x, y, 0^+) = \Phi(x, y, 0^-), \quad \text{for } (x, y) \in S. \tag{12}$$

In view of symmetry of the problem with respect to the crack surface, it is equivalent here to solve a mixed–mixed boundary value problem for the half-space $z \geq 0$, subject to the following boundary conditions on the plane $z = 0$:

$$\begin{array}{ll} \sigma_z = -p(x, y), & \text{for } (x, y) \in S; \\ w = 0, & \text{for } (x, y) \notin S; \\ \Phi = 0, & \text{for } -\infty < (x, y) < \infty; \\ \tau_z = 0, & \text{for } -\infty < (x, y) < \infty. \end{array} \quad (13)$$

The continuity of D_z is implicit by the symmetry condition.

Compared to the one for an impermeable crack (Chen and Shioya, 1999a), the most significant difference is that no jump of electric potential takes place at the crack face in the current problem. Thus, the form of harmonic functions F_i should be different from those assumed in Chen and Shioya (1999a). We will begin our analysis by considering the case of distinct eigenvalues, which is then followed by the discussions on cases of equal eigenvalues.

3.1. Case for distinct s_i

Since $\Phi = 0$ at the crack plane, we can assume

$$F_i(x, y, z) = c_i G(x, y, z_i) \quad (i = 1, 2, 3), \quad F_4(x, y, z) = 0, \quad (14)$$

where c_i are constants to be determined, and

$$G(M) = \iint_S \frac{\omega(N)}{R(M, N)} dS, \quad (15)$$

represents the potential of a simple layer, here ω is the crack face displacement $w(x, y, 0)$, $R(M, N)$ is the distance between the points $M(x, y, z)$ and $N(\xi, \eta, 0)$, and the integration is taken over the crack region S. Compared to that for an impermeable crack (Chen and Shioya, 1999a), the functions $F_i (i = 1, 2, 3)$ are only represented by one potential of a simple layer. The physical meaning of canceling another potential of a simple layer, which corresponds to the electric field, is obvious since there is no discontinuity of electric potential across the crack. It is also interesting to note that Equation (14) is identical to the case of pure elasticity (Fabrikant, 1989), except that the index i now ranges from 1 to 3, while it takes 1 and 2 only in Fabrikant (1989).

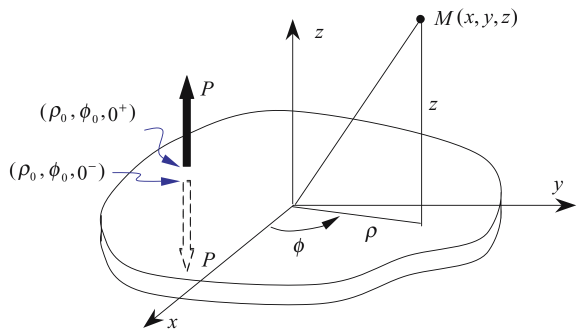

Figure 1. A flat crack in an infinite piezoelectric medium.

To satisfy the fourth condition in Equation (13), we take

$$\sum_{i=1}^{3} c_i \gamma_{i1} s_i = 0, \tag{16}$$

Using the property of potential of a simple layer and considering the second and third conditions in Equation (13), we arrive at

$$\sum_{i=1}^{3} c_i \alpha_{i1} = -\frac{1}{2\pi}, \quad \sum_{i=1}^{3} c_i \alpha_{i2} = 0. \tag{17}$$

Thus the constants c_i are completely determined as

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{11}s_1 & \gamma_{21}s_2 & \gamma_{31}s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}. \tag{18}$$

Considering the first condition in Equation (13) yields the following integro-differential equation

$$p(N_0) = -g_1 \Delta \iint_S \frac{\omega(N)}{R(N_0, N)} dS, \tag{19}$$

where, $g_1 = -\sum_{i=1}^{3} c_i \gamma_{i1}$, $R(N_0, N)$ is the distance between two points N_0 and N, and both N_0, $N \in S$. It is clear that Equation (19) still takes the same form as that reported in Fabrikant (1989). Hence, like the case of an impermeable crack, the splendid results obtained by Fabrikant (1989) can be used for our present purpose. The constant g_1 in Equation (19) is identical to that defined in Chen and Shioya (1999a).

3.2. Case for $s_1 \neq s_2 = s_3$

In this case, we should assume

$$F_i(x, y, z) = c_i G(x, y, z_i) \quad (i = 1, 2),$$
$$F_3(x, y, z) = c_3 H(x, y, z_2), \quad F_4(x, y, z) = 0, \tag{20}$$

where

$$H(M) = \frac{\partial G(M)}{\partial z}. \tag{21}$$

It is easy to show that the constants c_i are now determined by

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{11}s_1 & \gamma_{21}s_2 & \omega_{11} \\ \alpha_{11} & \alpha_{21} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{42} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}. \tag{22}$$

The corresponding governing equation is

$$p(N_0) = -g_1' \Delta \iint_S \frac{\omega(N)}{R(N_0, N)} \, \mathrm{d}S, \tag{23}$$

where, $g_1' = -\sum_{i=1}^{2} c_i\gamma_{i1} - c_3\gamma_{41}$. The above equation is the same as Equation (19) except the coefficient involved.

3.3. Case for $s_1 = s_2 = s_3$

In this case, we should assume

$$\begin{aligned} &F_1(x,y,z) = c_1 G(x,y,z_1),\\ &F_i(x,y,z) = c_i H(x,y,z_1) \quad (i=2,3),\\ &F_4(x,y,z) = 0. \end{aligned} \tag{24}$$

It is easy to show that the constants c_i are determined by

$$\begin{Bmatrix} c_1\\ c_2\\ c_3 \end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} \gamma_{11}s_1 & \omega_{11} & \omega_{21}\\ \alpha_{11} & \alpha_{41} & \alpha_{51}\\ \alpha_{12} & \alpha_{42} & \alpha_{52} \end{bmatrix}^{-1}\begin{Bmatrix} 0\\ 1\\ 0 \end{Bmatrix}. \tag{25}$$

The corresponding governing equation is

$$p(N_0) = -g_1''\Delta\iint\limits_S \frac{\omega(N)}{R(N_0,N)}\,\mathrm{d}S, \tag{26}$$

where, $g_1'' = -c_1\gamma_{11} - c_2\gamma_{41} - c_3\gamma_{51}$. The above equation is also the same as Equation (19) except the constant involved.

4. Exact Solution for a Penny-shaped Crack

In the following, we will focus on the case of distinct eigenvalues only. The derivation for equal eigenvalues is very similar because the structure of the governing equations is the same, as shown in the previous section. For the problem of a penny-shaped crack of radius a, the cylindrical coordinates (ρ,ϕ,z) is used for the sake of convenience. In this case, the analytical solution to Equation (19) can be obtained in integral form, and the reader is referred to Fabrikant (1989) for details. The substitution of this analytical solution into Equation (15) leads to the following expression for the potential G

$$G(\rho,\phi,z) = \frac{1}{2\pi^3 g_1}\int_0^{2\pi}\int_0^a K(\rho,\phi,z;\rho_0,\phi_0)\,p(\rho_0,\phi_0)\,\rho_0\,d\rho_0\,d\phi_0, \tag{27}$$

where K is called the Green's function and defined as (Fabrikant, 1989)

$$\begin{aligned} K(M;N_0) &= K(\rho,\phi,z;\rho_0,\phi_0)\\ &= \int_0^{2\pi}\int_0^a \frac{1}{R(N,N_0)}\tan^{-1}\left[\sqrt{\frac{(a^2-r^2)(a^2-\rho_0^2)}{aR(N,N_0)}}\right]\frac{r\,dr\,d\psi}{R(M,N)}, \end{aligned} \tag{28}$$

and $R(\cdot,\cdot)$ denotes the distance between respective points: $M(\rho,\phi,z)$, $N(r,\psi,0)$ and $N_0(\rho_0,\phi_0,0)$. Various derivatives of the Green's function K have been presented in Fabrikant (1989) and omitted here for brevity.

It is obvious now, according to Fabrikant (1989), all elastoelectric field variables can be expressed in elementary functions for any polynomial form distribution of p. This conclusion has also been reached for an impermeable crack (Chen and Shioya, 1999a). In the following two sub-sections, we will present the complete solution for point force loading as well as the one for uniform loading.

4.1. Exact point force solution

It is supposed that the penny-shaped crack is subjected to a pair of normal concentrated forces P applied in opposite directions at the points $(\rho_0, \phi_0, 0^{\pm})$, $\rho_0 < a$. By virtue of the analytical expressions for various derivatives of K (Fabrikant, 1989) as well as the property of δ-function, the following complete solution of the elastoelectric field is easily obtained

$$U = \frac{P}{g_1\pi^2}\sum_{i=1}^{3} c_i f_1(z_i), \quad w_k = -\frac{P}{g_1\pi^2}\sum_{i=1}^{3} \alpha_{ik} c_i f_2(z_i),$$
$$\sigma_{zj} = \frac{P}{g_1\pi^2}\sum_{i=1}^{3} \gamma_{ij} c_i f_3(z_i), \quad \sigma_2 = \frac{2c_{66}P}{g_1\pi^2}\sum_{i=1}^{3} c_i f_4(z_i),$$
$$\tau_{zk} = \frac{P}{g_1\pi^2}\sum_{i=1}^{3} \gamma_{ik} s_i c_i f_5(z_i), \tag{29}$$

where

$$f_1(z) = \frac{1}{\bar{t}}\left[\frac{z}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{\sqrt{a^2-\rho_0^2}}{\bar{s}}\tan^{-1}\left(\frac{\bar{s}}{\sqrt{l_2^2-a^2}}\right)\right],$$

$$f_2(z) = \frac{1}{R_0}\tan^{-1}\left(\frac{h}{R_0}\right),$$

$$f_3(z) = \frac{z}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) - \frac{h}{z(R_0^2+h^2)}\left(\frac{\rho^2-l_1^2}{l_2^2-l_1^2} - \frac{z^2}{R_0^2}\right),$$

$$f_4(z) = \frac{\sqrt{a^2-\rho_0^2}}{\bar{t}\bar{s}}\left(\frac{2}{\bar{t}} - \frac{\rho_0 e^{i\phi_0}}{\bar{s}^2}\right)\tan^{-1}\left(\frac{\bar{s}}{\sqrt{l_2^2-a^2}}\right) - \frac{z(3R_0^2-z^2)}{\bar{t}^2 R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right)$$
$$+\frac{\sqrt{a^2-\rho_0^2}\sqrt{l_2^2-a^2}\,\rho_0 e^{i\phi_0}}{\bar{t}\bar{s}^2[l_2^2-\rho\rho_0 e^{-i(\phi-\phi_0)}]} - \frac{zh}{R_0^2+h^2}\left[\frac{t}{\bar{t}R_0^2} - \frac{\rho^2 e^{2i\phi}}{(l_2^2-l_1^2)(l_2^2-\rho^2)}\right], \tag{30}$$

$$f_5(z) = \frac{t}{R_0^3}\tan^{-1}\left(\frac{h}{R_0}\right) + \frac{h}{R_0^2+h^2}\left(\frac{\rho e^{i\phi}}{l_2^2-l_1^2} + \frac{t}{R_0^2}\right),$$

in which

$$t=\rho e^{i\phi}-\rho_0 e^{i\phi_0},\quad \bar{s}=\sqrt{a^2-\rho\rho_0 e^{-i(\phi-\phi_0)}},\quad h=\sqrt{(a^2-l_1^2)(a^2-\rho_0^2)}/a,$$
$$R_0=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2},$$
$$l_1=\frac{1}{2}\left[\sqrt{(\rho+a)^2+z^2}-\sqrt{(\rho-a)^2+z^2}\right],\quad l_2=\frac{1}{2}\left[\sqrt{(\rho+a)^2+z^2}+\sqrt{(\rho-a)^2+z^2}\right]. \tag{31}$$

Noticing that

$$z=0:\quad l_1\to\min(a,\rho),\ \text{ and } l_2\to\max(a,\rho), \tag{32}$$

we can derive

$$z=0, r>a:\sigma_{zk}=-\frac{P}{g_1\pi^2}\left(\sum_{i=1}^{3}\gamma_{ik}c_i\right)\frac{1}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)}\sqrt{\frac{a^2-\rho_0^2}{\rho^2-a^2}}. \tag{33}$$

Defining the following stress and electric displacement intensity factors

$$k_\sigma=\lim_{\rho\to a}\left\{\sqrt{(\rho-a)}\sigma_z\Big|_{z=0}\right\},\quad k_D=\lim_{\rho\to a}\left\{\sqrt{(\rho-a)}D_z\Big|_{z=0}\right\}. \tag{34}$$

we obtain

$$k_\sigma=\frac{P}{\pi^2\sqrt{2a}}\frac{\sqrt{a^2-\rho_0^2}}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)},$$
$$k_D=\frac{\beta P}{\pi^2\sqrt{2a}}\frac{\sqrt{a^2-\rho_0^2}}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)}, \tag{35}$$

where $\beta=-\sum_{i=1}^{3}\gamma_{i2}c_i/g_1$. It is noted that the distribution of normal stress σ_z at the crack plane as well as the stress intensity factor (SIF) obtained here for a permeable crack are identical to those for an impermeable crack (Chen and Shioya, 1999a), indicating that the mechanical singularity behavior is not affected by the electric condition assumed on the crack faces. However, the electric field changes due to the different electric conditions employed and the electric displacement intensity factor (EDIF) in Equation (35) is much different from that obtained in Chen and Shioya (1999a). For the impermeable crack, the EDIF is independent of the applied mechanical load and the electric field singularity at the crack-tip is caused only by the electric load; the expression for EDIF is also independent of the material constants. For the permeable crack, the external mechanical load makes both the stress field and the electric displacement field singular at the crack-tip, and the expression for EDIF becomes dependent on the material constants through the electro-mechanical coupling parameter β introduced in Equation (35). Note that, for a permeable crack, the electric load applied at infinity does not induce the singularity of electric displacement at the crack-tip.

The SIF and EDIF for an arbitrarily distributed pressure p can be obtained by integrating Equation (35) over the crack domain, i.e.

$$
\begin{aligned}
k_\sigma &= \frac{1}{\pi^2\sqrt{2a}} \int_0^{2\pi}\int_0^a \frac{p(\rho_0,\phi_0)\sqrt{a^2-\rho_0^2}}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)}\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \\
k_D &= \frac{\beta}{\pi^2\sqrt{2a}} \int_0^{2\pi}\int_0^a \frac{p(\rho_0,\phi_0)\sqrt{a^2-\rho_0^2}}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)}\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0.
\end{aligned} \tag{36}
$$

4.2. Exact solution for uniform load

Now consider the penny-shaped crack subjected to two equal and opposite uniform pressures p_0 symmetrically applied to the upper and lower crack faces. From Fabrikant (1989), the exact solution to Equation (19) is known to be

$$
\omega = \frac{p_0}{g_1\pi^2}\sqrt{a^2-\rho^2} \quad (\rho \leqslant a). \tag{37}
$$

Using the above solution, we obtain from Equations (15), (14), (2) and (4)

$$
\begin{aligned}
U &= -\frac{p_0}{g_1\pi}\rho e^{i\phi}\sum_{i=1}^3 c_i\left[\frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2} - \sin^{-1}\left(\frac{a}{l_{2i}}\right)\right], \\
w_k &= \frac{2p_0}{g_1\pi}\sum_{i=1}^3 \alpha_{ik}c_i\left[z_i\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \sqrt{a^2-l_{1i}^2}\right], \\
\sigma_{zj} &= \frac{2p_0}{g_1\pi}\sum_{i=1}^3 \gamma_{ij}c_i\left[\sin^{-1}\left(\frac{a}{l_{2i}}\right) - \frac{a\sqrt{l_{2i}^2-a^2}}{l_{2i}^2-l_{1i}^2}\right], \\
\sigma_2 &= -\frac{4c_{66}p_0}{g_1\pi}ae^{i2\phi}\sum_{i=1}^3 c_i\frac{l_{1i}^2\sqrt{l_{2i}^2-a^2}}{l_{2i}^2(l_{2i}^2-l_{1i}^2)}, \\
\tau_{zk} &= \frac{2p_0}{g_1\pi}a^2\rho e^{i\phi}\sum_{i=1}^3 \gamma_{ik}s_ic_i\frac{\sqrt{a^2-l_{1i}^2}}{l_{2i}^2(l_{2i}^2-l_{1i}^2)},
\end{aligned} \tag{38}
$$

where

$$
l_{1i} = \frac{1}{2}\left[\sqrt{(\rho+a)^2+z_i^2} - \sqrt{(\rho-a)^2+z_i^2}\right], \quad l_{2i} = \frac{1}{2}\left[\sqrt{(\rho+a)^2+z_i^2} + \sqrt{(\rho-a)^2+z_i^2}\right]. \tag{39}
$$

Noticing Equation (32), we obtain from Equation (38)

$$
z=0,\ \rho<a:\ \sigma_{zk} = \frac{p_0}{g_1}\left(\sum_{i=1}^3 \gamma_{ik}c_i\right) = \begin{cases} -p_0 & \text{for } k=1, \\ -\beta p_0 & \text{for } k=2, \end{cases}
$$

$$z=0, r>a: \sigma_{zk}=\frac{2p_0}{g_1\pi}\left(\sum_{i=1}^{3}\gamma_{ik}c_i\right)\left[\sin^{-1}\left(\frac{a}{\rho}\right)-\frac{a}{\sqrt{\rho^2-a^2}}\right]$$
$$=\begin{cases} -\frac{2p_0}{\pi}\left[\sin^{-1}\left(\frac{a}{\rho}\right)-\frac{a}{\sqrt{\rho^2-a^2}}\right] & \text{for } k=1, \\ -\frac{2\beta p_0}{\pi}\left[\sin^{-1}\left(\frac{a}{\rho}\right)-\frac{a}{\sqrt{\rho^2-a^2}}\right] & \text{for } k=2. \end{cases} \tag{40}$$

Obviously, the electric displacement does not vanish when the permeable electric conditions are assumed on the crack surfaces. The SIF and EDIF for the case of uniform loading are obtained as

$$k_\sigma=\frac{p_0}{\pi}\sqrt{2a}, \quad k_D=\frac{\beta p_0}{\pi}\sqrt{2a}. \tag{41}$$

The above formulations can also be obtained by setting $p=p_0$=constant in Equation (36) and performing the integration.

Yang and Lee (2001) obtained the intensity factors for the axisymmetric problem of a penny-shaped crack embedded in a piezoelectric strip. When the thickness of the strip tends to infinity, the intensity factors derived by them [setting $\Psi(1)=1$ in Equation (50) in Yang and Lee (2001), for load case (1)] are found exactly the same as those presented in Equation (41).

4.3. Numerical discussion

As shown in Equations (36), the electro-mechanical coupling parameter β plays a very important role in interpreting the singular behavior of the electric displacement field due to a mechanical load. Table 1 shows its values for ten different engineering piezoelectric materials that characterizes transverse isotropy. The material constants for these piezoelectric materials can be found in the references indicated in the table, and are not repeated here for brevity. For the sake of completeness, the eigenvalues $s_{i=1,2,3}$, are also presented in Table 1.

The axial displacement w at the crack face is equal to half of the crack opening displacement (COD), which is an important index in fracture analysis. In Chen and Shioya (1999a), the distribution curves of the nondimensional crack surface displacement were given when the impermeable crack is subjected to a pair of concentrated forces P for the piezoelectric ceramic PZT-6B. These curves can be directly scaled and converted to those for the permeable crack by noticing the following relationship:

$$\kappa=\frac{w_p}{w_i}=\frac{1}{4\pi^2 g_1 g_4 A}, \tag{42}$$

where g_4 and A are constants defined in Chen and Shioya (1999a). The subscripts p and i indicate permeable and impermeable conditions, respectively. The COD ratio κ, between permeable and impermeable cracks defined above can also serve as an important parameter for understanding the effect of electric conditions, which are assumed along the crack faces, on the fracture behavior of piezoelectric materials. Note that this formulation is valid for an arbitrary flat crack subjected to an arbitrarily distributed mechanical load. The values of κ for different materials have been

Table 1. Electro-mechanical coupling parameter and COD ratio.

Reference	Material	Eigenvalues s_i	Coupling parameter β	COD ratio κ
Dunn and Taya (1994)	PZT-4	1.06914 − 0.200381i 1.06914 + 0.200381i 1.20380	2.50098×10^{-10}	1.31147
	PZT-5	1.07735 − 0.248490i 1.07735 + 0.248490i 1.07766	3.03516×10^{-10}	1.30938
	PZT-7A	1.02040 − 0.0698338i 1.02040 + 0.0698338i 1.70258	1.29782×10^{-10}	1.19872
	$BaTiO_3$	0.940505 1.00382 − 0.229193i 1.00382 + 0.229193i	1.66785×10^{-10}	1.14414
Royer and Dieulesaint (2000)	ZnO	0.635143−0.299172 i 0.635143 + 0.299172i 1.99317	1.58502×10^{-12}	1.00232
	CdS	0.697279 − 0.0887886i 0.697279 + 0.0887886i 1.89471	1.43325×10^{-12}	1.00072
Wang and Zheng (1995)	PZT-6B	0.517601 1.01433 2.10146	7.10604×10^{-11}	1.07698
Bisegna and Maceri (1996)	PZT-5H	1.02925 − 0.414981i 1.02925+0.414981 i 1.05333	4.28021×10^{-10}	1.33835
	PZT-8	1.04378 − 0.241853i 1.04378 + 0.241853i 1.19534	1.77408×10^{-10}	1.18502
	C-24	1.03234 − 0.127467i 1.03234 + 0.127467i 1.06493	6.72124×10^{-11}	1.17144

calculated and are presented in Table 1. It is seen that the COD ratio κ is generally greater than unity, which is reasonable since the impermeable electric condition allows a simultaneous jump in electric potential, in addition to the COD, when a mechanical load is applied.

It can be seen that, among these materials, the electro-mechanical coupling parameter β of PZT-5H is the largest, indicating that the EDIF of PZT-5H is the most prominent. Generally, the engineering piezo-ceramics have a relatively stronger coupling between the elastic deformation and electric field, leading to a larger electro-mechanical coupling parameter. However, a large β does not imply that the material is susceptible to the electric failure due to crack, since the onset of cracking should also depend on the electric fracture toughness of the specific material.

5. Conclusions

The method of potential theory is employed to analyze the permeable crack problem of transversely isotropic piezoelectric materials. The analysis is confined to the particular case that the crack plane is parallel to the plane of isotropy. When compared to the analysis for an impermeable crack, the harmonic functions in the general solution do not contain the potential of a simple layer, which corresponds to the electric field and was introduced by Chen and Shioya (1999a). This is a natural result owing to the continuity of electric potential at the crack face imposed by the permeable electric condition.

For a penny-shaped crack, exact solutions for three-dimensional point force and that for a uniform loading are obtained, and the elasto-electric field variables in the full-space are expressed in terms of elementary functions. It should be noted that the point force solution is very important because it can be used to construct other important analytical solutions for other types of loadings, and can also serve as a fundamental solution in BEM analysis of a finite cracked body.

The SIF and EDIF are derived in a very simple form. Unlike the impermeable crack, the applied mechanical load causes both the singularity of stress field and the singularity of electric displacement field at the crack tip of a permeable crack. There is no singularity either in the stress filed or in the electric displacement field if the mechanical load vanishes. The SIF is independent of the material constants and is the same as that for an elastic isotropic material, and it agrees well with Kogan et al. (1996). The EDIF is proportional to the SIF by an electro-mechanical coupling parameter β, which is a constant for a specific material and does not vary with the crack shape and the load distribution. The COD of a permeable crack is also related to the COD of an impermeable crack by another constant κ. This constant is generally larger than unity, indicating that the COD of a permeable crack is more significant than that of an impermeable crack. Thus, unlike the SIF based fracture criterion, the CTOD (crack tip opening displacement) based fracture criterion will be certainly affected by the electric conditions at the crack surface.

The general solutions for equal eigenvalues are presented in the paper. By assuming a proper form of harmonic functions, the resulting governing equation is exactly the same as that for the case of distinct eigenvalues except the constant involved. Thus, the corresponding solutions for equal eigenvalues can be easily derived. This method is different from the method based on the L'Hospital rule as suggested by Fabrikant (1989). It is noted that, since more independent material constants are involved, the use of L'Hospital rule will become more cumbersome for piezoelectric materials.

Finally, the paper emphasizes the particular case when an external mechanical force is applied directly to the crack faces. The solution for the presence of both mechanical and electric loads at places other than the crack face can be easily achieved by the principle of superposition. In this general case, the complete solution consists of two parts: (i) the first part corresponding to a perfect medium under the action of external loads without the disturbance of crack, and (ii) the second part by setting the external loads on the crack surfaces equal and opposite to the stresses induced at the crack site in the solution of the first part. Note that, if mechanical displacements (or strains) and electric potential (or electric field) are prescribed at infinity, stresses will be generally induced at the crack site, indicating that singularity will emerge at the crack tip both in stress field and electric displacement field.

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