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A LIPSCHITZ ESTIMATE FOR MULTILINEAR OSCILLATORY SINGULAR INTEGRALS WITH ROUGH KERNELS [1]

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Abstract In this paper, for the multilinear oscillatory singular integral operators $T_{A_1,A_2,\cdots,A_r}$ defined by

$$T_{A_1,A_2,\cdots,A_r}f(x)=\text{p.v.}\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^{r}R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y,\quad n\geqslant 2,$$

where $P(x,y)$ is a nontrivial and real-valued polynomial defined on $\mathbf{R}^n\times\mathbf{R}^n$, $\Omega(x)$ is homogeneous of degree zero on $\mathbf{R}^n$, $A_s(x)$ has derivatives of order m_s in $\dot{\Lambda}_{\beta_s}$ $(0<\beta_s<1)$, $R_{m_s+1}(A_s;x,y)$ denotes the (m_s+1)-st remainder of the Taylor series of A_s at x expended about y $(s=1,2,\cdots,r)$, $M=\sum_{s=1}^r m_s$, the author proves that if $0<\beta=\sum_{s=1}^r\beta_s<1$, and $\Omega\in L^q(S^{n-1})$ for some $q>1/(1-\beta)$, then for any $p\in(1,\infty)$, and some appropriate $0<\beta<1$, $T_{A_1,A_2,\cdots,A_r}$ is bounded on $L^p(\mathbf{R}^n)$.

Key words Multilinear operator, oscillatory singular integral, Lipschitz spaces, rough kernel

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1 Introduction

As well-known, oscillatory singular integral operators with polynomial phase are very useful in the study of Hilbert transforms along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms. There has been significant progress in the study of this type of operators since Ricci and Stein[14] gave the prototypical work in this area (see [14,2,12,10,8] et al and their references). Here we consider the following multilinear oscillatory singular integral operator defined by

$$T_{A_1,A_2,\cdots,A_r}f(x)=\text{p.v.}\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^{r}R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y,\quad n\geqslant 2,$$

where r, m_s $(s=1,2,\cdots,r)$ are positive numbers, $M=\sum_{s=1}^r m_s$, $P(x,y)$ is a real polynomial on $\mathbf{R}^n$, Ω is homogeneous of degree zero on $\mathbf{R}^n$ and $\int_{S^{n-1}}\Omega(x')\mathrm{d}x'=0$, S^{n-1} denotes the

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unit sphere of $\mathbf{R}^n$, $A_s(x)$ has derivatives of order m_s in $\mathbf{R}^n$, $R_{m_s+1}(A_s;x,y)$ is the (m_s+1)-st $(m_s\geqslant 1)$ order remainder of the Taylor series of A_s expended at x about y, precisely,

$$R_{m_s+1}(A_s;x,y)=A_s(x)-\sum_{|\gamma|\leqslant m}\frac{1}{\gamma!}D^\gamma A_s(y)(x-y)^\gamma.$$

For operators of this type, there have been many interesting works (see [4,5,6,9] et al). Here , we consider the case that $D^\gamma A_s\in\dot{\Lambda}_{\beta_s}(\mathbf{R}^n)$ $(|\gamma|\leqslant m_s)$, where $\dot{\Lambda}_{\beta_s}$ denotes the Lipschitz space defined by

$$\dot{\Lambda}_{\beta_s}(\mathbf{R}^n)=\left\{f:\quad \|f\|_{\dot{\Lambda}_{\beta_s}}=\sup_{x,\,h\in\mathbf{R}^n;\,h\neq 0}\frac{|\Delta_h^{[\beta_s]+1}f(x)|}{|h|^{\beta_s}}<\infty\right\},\qquad s=1,2,\cdots,r,$$

where $\Delta_h^1 f(x)=f(x+h)-f(x)$, $\Delta_h^{k+1}f(x)=\Delta_h^1(\Delta_h^{k-1}f)(x)$. It is easy to see that, for $s\in\{1,\,2,\,\cdots,r\}$, if $0<\beta_s<1$, $f(x)\in\dot{\Lambda}_{\beta_s}$, then

$$|f(x)-f(y)|\leqslant|x-y|^{\beta_s}\|f\|_{\dot{\Lambda}_{\beta_s}},\qquad\forall\ \ x,\,y\in\mathbf{R}^n.\tag{1.1}$$

When $r=1$, we denote A_1 by A, β_1 by β, m_1 by m, and T_{A_1} by T_A. For the corresponding multilinear operator related to singular integral defined by

$$\overline{T}_Af(x)=\text{p.v.}\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+m}}R_{m+1}(A;\,x,y)f(y)\mathrm{d}y,$$

Chen[3] shown that if $\Omega\in\mathrm{Lip}_1(S^{n-1})$, then for $1/r=1/p-\beta/n$,

$$\|\overline{T}_Af\|_p\leqslant C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\,\|f\|_p.$$

Recently, Lu, the author, and Zhang[11] improved the above result to the case $\Omega\in L^q(S^{n-1})$ for some $q\geqslant n/(n-\beta)$. These results indicates that for $D^\gamma A\in\dot{\Lambda}_\beta$ $(|\gamma|=m)$, $\overline{T}_A$ has the same mapping properties on the Lebesgue spaces as those of the fractional integral operator $\overline{T}$ defined by

$$\overline{T}f(x)=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\beta}}f(y)\mathrm{d}y.$$

It is naturally led to the question whether T_A has the same mapping properties on L^p as those of the fractional oscillatory integral operator. In this paper, we will consider the more general operator $T_{A_1,A_2,\cdots,A_r}$.

For the fractional oscillatory singular integral operator with smoothness kernel, Ricci and Stein[14] showed the following result.

Theorem A[14] For each $d\geqslant 2$, there exists an $a_d>0$, such that whenever (i) $P(x,y)$ is a real polynomial of total degree $\leqslant d$, which is nontrivial in the sense that it cannot be written as $P_0(x)+P_1(y)$, and (ii) $K(x,y)$ is a function which satisfies $|K(x,y)|\leqslant C|x-y|^{-n+\beta}$, $|\nabla K(x,y)|\leqslant C|x-y|^{-n+\beta-1}$, then the operator T defined by

$$Tf(x)=\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}P(x,y)}K(x,y)f(y)\mathrm{d}y$$

is bounded on $L^p(\mathbf{R}^n)$, where $0<\beta<a_d(1/2-|1/p-1/2|)$, and the bound of the operator do depend on the polynomial $P(x,y)$.

In 1996, Y. Ding[8] improved the above result as follows.

Theorem B[8] Suppose that Ω is homogeneous of degree zero on $\mathbf{R}^n$ and belongs to $L^q(S^{n-1})$ for some $q>1$, $b(r)\in BV(\mathbf{R}_+)$, $P(x,y)=\sum_{|\xi|\leqslant k,|\eta|\leqslant l}a_{\xi\eta}x^\xi y^\eta$ is a nontrivial polynomial on $\mathbf{R}^n\times\mathbf{R}^n$. Then for the fractional oscillatory singular integral operator

$$Tf(x)=\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n-\beta}}b(|x-y|)f(y)\mathrm{d}y,$$

(i) if $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}$ and $q>1/(1-\beta)$, then T is bounded on $L^2(\mathbf{R}^n)$;

(ii) if $1<p<\infty$ $(p\neq 2)$, $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}\{1/2-|1/p-1/2|\}$ and $q>1/(1-\beta)$, then T is bounded on $L^p(\mathbf{R}^n)$. Here the bound of T depend on the value of $\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|$, but not on the other coefficients of $P(x,y)$.

The purpose of this paper is to show that $T_{A_1,A_2,\cdots,A_r}$ enjoys some properties, which are parallel to those of the fractional oscillatory integral operator T, and to give a positive answer to the above question. Our main result can be stated as follows.

Theorem 1 Suppose that Ω, $R_{m_s+1}(A_s;x,y)$ is as above, $D^\gamma A_s\in\dot{\Lambda}_{\beta_s}$ $(|\gamma|=m_s)$, $s\in\{1,2,\cdots,r\}$, $\beta=\sum_{s=1}^r\beta_s$, $P(x,y)=\sum_{|\xi|\leqslant k,|\eta|\leqslant l}a_{\xi\eta}x^\xi y^\eta$ is a nontrivial polynomial on $\mathbf{R}^n\times\mathbf{R}^n$. Then for $\Omega\in L^q(S^{n-1})$,

(i) if $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}$ and $q>1/(1-\beta)$, then

$$\|T_{A_1,A_2,\cdots,A_r}f\|_2\leqslant C(n,a,M,\deg P)\prod_{s=1}^r\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|f\|_2.$$

(ii) if $1<p<\infty$ $(p\neq 2)$, $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}\{1/2-|1/p-1/2|\}$ and $q>1/(1-\beta)$, then

$$\|T_{A_1,A_2,\cdots,A_r}f\|_p\leqslant C(n,a,M,\deg P)\prod_{s=1}^r\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|f\|_p.$$

Here $a=(\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|)^{1/(k+l)}$, $\deg P$ denotes the total degree of $P(x,y)$.

Remark 1 It is worth pointing out that the bound of the fractional oscillatory operators in Theorem A do depend on the polynomial $P(x,y)$, but the bound of $T_{A_1,A_2,\cdots,A_r}$ in our theorem, as those of T in Theorem B, depend only on the value of $\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|$ but not on the other coefficients of $P(x,y)$.

When $r=1$, we have the corresponding result as follows.

Theorem 2 Suppose that Ω, $R_{m+1}(A;x,y)$ is as above, $D^\gamma A\in\dot{\Lambda}_\beta$ $(|\gamma|=m)$, $P(x,y)=\sum_{|\xi|\leqslant k,|\eta|\leqslant l}a_{\xi\eta}x^\xi y^\eta$ is a nontrivial polynomial on $\mathbf{R}^n\times\mathbf{R}^n$ and a is as in Theorem 1. Then for $\Omega\in L^q(S^{n-1})$,

(i) if $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}$ and $q>1/(1-\beta)$, then

$$\|T_Af\|_2\leqslant C(n,a,m,\deg P)\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\|f\|_2.$$

(ii) if $1<p<\infty$ $(p\neq 2)$, $0<\beta<\min\{(l+k)/2k,\ (l+k)/2l\}\{1/2-|1/p-1/2|\}$ and $q>1/(1-\beta)$, then $\|T_Af\|_p\leqslant C(n,a,m,\deg P)\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\|f\|_p$.

Here $a=(\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|)^{1/(k+l)}$, $\deg P$ denotes the total degree of $P(x,y)$.

This paper is organized as follows. In Section 2, we will give some preliminary lemmas. The proofs of our theorems will be given in Section 3. We would remark that our some ideas in the proofs of our theorems are taken from [14,9,8]. Throughout the rest of this paper, we always use the letter C to denote positive constants that may vary at each occurrence, but is independent of the essential variables.

2 Some Lemmas

In this section, we give some preliminary lemmas.

Lemma 1[11] Let Q be a cube centered at x with diameter r. If $D^\gamma A \in \dot{\Lambda}_\beta$ $(0<\beta<1, |\gamma|=m)$, then for $|x-y|<r$, $|R_{m+1}(A;x,y)| \leqslant Cr^\beta |x-y|^m \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}$.

Lemma 2[7] Let $A(x)$ be a function on $\mathbf{R}^n$ with m-th order derivatives in $L^t_{loc}(\mathbf{R}^n)$ for some $t>n$. Then

$$|R_m(A;x,y)| \leqslant C_{m,n}|x-y|^m \sum_{|\gamma|=m}\left(\frac{1}{|Q_x^y|}\int_{Q_x^y}|D^\gamma A(z)|^t\mathrm{d}z\right)^{1/t},$$

where Q_x^y is the cube centered at x with diameter $5\sqrt{n}|x-y|$.

Lemma 3[13] Let $0<\beta<1, 1\leqslant q<\infty$, we have

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{Q}\int_Q |f(x)-m_Q(f)|^q\mathrm{d}x\right)^{1/q},$$

where $m_Q(f)=1/|Q|\int_Q f(x)\mathrm{d}x$. For $q=\infty$, the formula should be interpreted appropriately.

Lemma 4[13] Let $Q^*\subset Q$, $g\in\dot{\Lambda}_\beta$ $(0<\beta<1)$. Then $|m_{Q^*}(g)-m_Q(g)|\leqslant C|Q|^{\beta/n}\|g\|_{\dot{\Lambda}_\beta}$.

3 Proofs of Theorems

We only prove Theorem 1 since the proof of Theorem 2 is similar. For simplicity, we only consider the case: $r=2$. Let us first consider the case $\sum_{|\xi|=k,|\eta|=l}|a_{\xi\eta}|=1$. Write

$$\begin{aligned} T_{A_1,A_2}f(x) &= \int_{|x-y|<1} \mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^2 R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y \\ &\quad + \int_{|x-y|\geqslant 1} \mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^2 R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y \\ &:= T^0_{A_1,A_2}f(x)+T^\infty_{A_1,A_2}f(x). \end{aligned} \tag{3.1}$$

At first, we estimate $\|T^0_{A_1,A_2}f\|_p$, $1<p<\infty$. By Lemma 1, we have

$$\begin{aligned} |T^0_{A_1,A_2}f(x)| &\leqslant \int_{|x-y|<1}\frac{|\Omega(x-y)|}{|x-y|^{n+M}}\prod_{s=1}^2|R_{m_s+1}(A_s;x,y)||f(y)|\mathrm{d}y \\ &= \sum_{j=0}^\infty\int_{2^{-j-1}\leqslant|x-y|<2^{-j}}\frac{|\Omega(x-y)|}{|x-y|^{n+M}}\prod_{s=1}^2|R_{m_s+1}(A_s;x,y)||f(y)|\mathrm{d}y \\ &\leqslant C\prod_{s=1}^2\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\sum_{j=0}^\infty 2^{-j\beta}\int_{2^{-j-1}\leqslant|x-y|<2^{-j}}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\sum_{j=0}^{\infty}2^{-j\beta}2^{jn}\int_{|x-y|<2^{-j}}|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\sum_{j=0}^{\infty}2^{-j\beta}M_\Omega f(x)\\
&\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}M_\Omega f(x),
\end{aligned}
$$

where M_Ω is the maximal operator with rough kernel defined by

$$M_\Omega f(x)=\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\mathrm{d}y.$$

Since $\Omega\in L^q(S^{n-1})\,(q>1)$, it follows from [1] that for any $1<p<\infty$, $\|M_\Omega f\|_p\leqslant C\|f\|_p$.

Thus, for $1<p<\infty$,

$$\|T^0_{A_1,A_2}f\|_p\leqslant C\prod_{s=1}^{r}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|M_\Omega f\|_p\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|f\|_p, \tag{3.2}$$

where C is independent of the coefficients of $P(x,y)$. It remains to show that

$$\|T^\infty_{A_1,A_2}f\|_p\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|f\|_p. \tag{3.3}$$

By letting

$$T^j_{A_1,A_2}f(x)=\int_{2^j\leqslant|x-y|<2^{j+1}}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^{2}R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y,$$

we have

$$T^\infty_{A_1,A_2}f(x)=\sum_{j=0}^{\infty}T^j_{A_1,A_2}f(x).$$

Obviously, in order to obtain (3.3), we only need to prove that there is a constant $\theta>0$ such that for every $1\leqslant j<\infty$,

$$\|T^j_{A_1,A_2}f\|_p\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f\|_p, \tag{3.4}$$

where C is independent of f and j.

We turn our attention to the operator

$$\widetilde{T}^j_{A_1,A_2}f(x)=2^{j\beta}\int_{1\leqslant|x-y|<2}\mathrm{e}^{\mathrm{i}P(2^jx,2^jy)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^{2}R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y.$$

It is easy to check up that the proof of (3.4) can be reduced to show that

$$\|\widetilde{T}^j_{A_1,A_2}f\|_p\leqslant C2^{-\theta j}\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\|f\|_p. \tag{3.5}$$

For fixed $j \in \mathbf{N}$, we now prove (3.5). Write $\mathbf{R}^n = \cup_d Q_d$, where each Q_d is a cube with side length 1 and the cube have disjoint interiors. Set $f_d = f\chi_{Q_d}$. Since that the support of $\widetilde{T}^j_{A_1,A_2} f_d$ is contained in a fixed multiple of Q_d, so the supports of various terms $\widetilde{T}^j_{A_1,A_2} f_d$ have bounded overlaps. Thus

$$\|\widetilde{T}^j_{A_1,A_2} f\|_p^p \leqslant C \sum_d \|\widetilde{T}^j_{A_1,A_2} f_d\|_p^p. \tag{3.6}$$

For each fixed d, denote $\overline{Q}_d = 10nQ_d$. From [7] we can take $\varphi_d(x) \in C_0^\infty(\mathbf{R}^n)$ such that $0 \leqslant \varphi_d \leqslant 1$, φ_d is identically one on $4\sqrt{n}Q_d$ and vanishes outside of $6\sqrt{n}Q_d$, $\|D^\gamma \varphi_d\|_\infty \leqslant C|\overline{Q}_d|^{-|\gamma|/n}$ for all multi-index γ $(|\gamma| \leqslant m)$.

Let x_0 be a point on the boundary of $8\sqrt{n}Q_d$. Denote

$$A_s^{Q_d}(y) = A_s(y) - \sum_{|\alpha_s|=m_s} \frac{1}{\alpha_s!} m_{\overline{Q}_d}(D^{\alpha_s} A_s) y^{\alpha_s},$$
$$A_s^{\varphi_d}(y) = R_{m_s}(A_s^{Q_d}; y, x_0)\varphi_d(y), \qquad s = 1,\, 2,$$

and for multi-index α,

$$\widetilde{T}^\alpha_j h(x) = 2^{j\beta} \int_{1 \leqslant |x-y| < 2} \mathrm{e}^{\mathrm{i}P(2^j x, 2^j y)} \frac{\Omega(x-y)}{|x-y|^{n+M}} (x-y)^\alpha h(y)\mathrm{d}y.$$

Noting that $R_{m_s+1}(A_s; x, y) = R_{m_s+1}(A_s^{\varphi_d}; x, y)$ for $x \in 4Q_d$ and $y \in Q_d$, it is easy to deduce that

$$\begin{aligned}
\widetilde{T}^j_{A_1,A_2} f_d(x) &= \widetilde{T}^j_{A_1^{\varphi_d},A_2^{\varphi_d}} f_d(x) \\
&= A_1^{\varphi_d}(x) A_2^{\varphi_d}(x) \widetilde{T}^0_j f(x) - A_1^{\varphi_d}(x) \sum_{0<|\alpha_2|<m_2} \frac{1}{\alpha_2!} \widetilde{T}^{\alpha_2}_j (D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&\quad - A_1^{\varphi_d}(x) \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \widetilde{T}^{\alpha_2}_j (D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&\quad - A_2^{\varphi_d}(x) \sum_{0<|\alpha_1|<m_1} \frac{1}{\alpha_1!} \widetilde{T}^{\alpha_1}_j (D^{\alpha_1} A_1^{\varphi_d} f_d)(x) \\
&\quad - A_2^{\varphi_d}(x) \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \widetilde{T}^{\alpha_1}_j (D^{\alpha_1} A_1^{\varphi_d} f_d)(x) \\
&\quad + \sum_{0<|\alpha_1|<m_1} \sum_{0<|\alpha_2|<m_2} \frac{1}{\alpha_1!\alpha_2!} \widetilde{T}^{\alpha_1+\alpha_2}_j (D^{\alpha_1} A_1^{\varphi_d} D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&\quad + \sum_{0<|\alpha_1|<m_1} \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \widetilde{T}^{\alpha_1+\alpha_2}_j (D^{\alpha_1} A_1^{\varphi_d} D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&\quad + \sum_{|\alpha_1|=m_1} \sum_{0<|\alpha_2|<m_2} \frac{1}{\alpha_1!\alpha_2!} \widetilde{T}^{\alpha_1+\alpha_2}_j (D^{\alpha_1} A_1^{\varphi_d} D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&\quad + \sum_{|\alpha_1|=m_1} \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \widetilde{T}^{\alpha_1+\alpha_2}_j (D^{\alpha_1} A_1^{\varphi_d} D^{\alpha_2} A_2^{\varphi_d} f_d)(x) \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.
\end{aligned}$$

To estimate these nine terms, we shall use the following lemma:

Lemma 5 Under the assumptions of Theorem 1, there exists a positive constant $\delta = \delta(n, \deg P)$ such that for any multi-index α and $j \geqslant 0$,

(i) $\beta-\delta-/\delta lk<0$ and $\delta<\min\{k/2l,\ k/q'(k+l)\}$ and

$$\|\widetilde{T}_j^{\alpha}h\|_2\leqslant C2^{(\beta-\delta-\delta l/k)j}\|h\|_2; \tag{3.7}$$

(ii) $\beta-\delta\sigma-\delta\sigma l/k<0$ and $\delta<\min\{1/2,\ k/2l,\ k/q'\sigma(k+l)\}$ and

$$\|\widetilde{T}_j^{\alpha}h\|_p\leqslant C2^{(\beta-\delta\sigma-\delta\sigma l/k)j}\|h\|_p, \tag{3.8}$$

where $1<p<\infty$ $(p\neq 2)$, $\sigma=1/2-|1/p-1/2|$. Here C is independent of the coefficients of $P(x,y)$.

Proof Let $b(r)=r^{|\alpha|-M}$ and $\overline{\Omega}(x)=\Omega(x)(x/|x|)^{\alpha}$. It is easy to see that $\overline{\Omega}(x)$ is homogeneous of degree zero and belongs to $L^q(S^{n-1})$. Note that

$$\begin{aligned}&2^{j\beta}\int_{2^{j-1}\leqslant|x-y|<2^j}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+M}}(x-y)^{\alpha}h(y)\mathrm{d}y\\ =\ &2^{j\beta}\int_{2^{j-1}\leqslant|x-y|<2^j}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\overline{\Omega}(x-y)}{|x-y|^{n}}b(|x-y|)h(y)\mathrm{d}y.\end{aligned}$$

Checking the argument of Ding in [8,p.73-78], we can find that there exists a positive constant $\delta=\delta(n,\deg P)$ such that

(i) $\beta-\delta-\delta l/k<0$ and $\delta<\min\{k/2l,\ k/q'(k+l)\}$ and

$$\left\|2^{j\beta}\int_{2^{j}\leqslant|x-y|<2^{j+1}}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\overline{\Omega}(x-y)}{|x-y|^{n}}b(|x-y|)h(y)\mathrm{d}y\right\|_2\leqslant C2^{(\beta-\delta-\delta l/k+|\alpha|-M)j}\|h\|_2;$$

(ii) $\beta-\delta\sigma-\delta\sigma l/k<0$ and $\delta<\min\{1/2,\ k/2l,\ k/q'\sigma(k+l)\}$ and

$$\left\|2^{j\beta}\int_{2^{j}\leqslant|x-y|<2^{j+1}}\mathrm{e}^{\mathrm{i}P(x,y)}\frac{\overline{\Omega}(x-y)}{|x-y|^{n}}b(|x-y|)h(y)\mathrm{d}y\right\|_p\leqslant C2^{(\beta-\delta\sigma-\delta\sigma l/k+|\alpha|-M)j}\|h\|_p,$$

where $1<p<\infty$ $(p\neq 2)$, $\sigma=1/2-|1/p-1/2|$. Here C is independent of the coefficients of $P(x,y)$. This leads to the conclusion of Lemma 5.

We now return to the proof of Theorem 1. Let α_s be a multi-index such that $|\alpha_s|\leqslant m_s$ $(s=1,\ 2)$, a straightforward computation (see [7,p.452]) yields that

$$D^{\alpha_s}A_s^{\varphi_d}(y)=\sum_{\alpha_s=\mu+\nu}C_{\mu,\nu}R_{m_s-|\mu|}(D^{\mu}A_s^{Q_d};y,x_0)D^{\nu}\varphi_d(y),\qquad s=1,\ 2. \tag{3.9}$$

Recall that $\mathrm{supp}\varphi_d\subset 6\sqrt{n}Q_d$, we can get by Lemma 2-4 that if $|\alpha_s|<m_s$,

$$\begin{aligned}|D^{\alpha_s}A_s^{\varphi_d}(y)|\leqslant&\sum_{\alpha_s=\mu+\nu}C_{\mu,\nu}|y-x_0|^{m_s-|\mu|}|\overline{Q}_d|^{-|\nu|/n}\\ &\times\sum_{|\eta|=m_s-|\mu|}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}|D^{\eta}(D^{\mu}A_sQ_d)(z)|^t\mathrm{d}z\right)^{1/t}\\ \leqslant&\ C|Q_d|^{(m_s-|\alpha_s|)/n}\sum_{|\alpha_s|=m_s}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}|D^{\alpha_s}A_s^{Q_k}(z)|^t\mathrm{d}z\right)^{1/t}\\ \leqslant&\ C\sum_{|\alpha_s|=m_s}\left(\frac{1}{|Q_y^{x_0}|}\int_{Q_y^{x_0}}\left|D^{\alpha_s}A_s(z)-m_{\overline{Q}_d}(D^{\alpha_s}A_s)\right|^t\mathrm{d}z\right)^{1/t}\\ \leqslant&\ C|\overline{Q}_d|^{\beta_s/n}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}\leqslant C\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}},\end{aligned}$$

where $n < t < \infty$.

If $|\alpha_s| = m_s$, by (3.9) and Lemma 2-4, we have

$$\begin{aligned}
|D_s^{\alpha_s} A_s^{\varphi_d}(y)| &\leqslant \sum_{\alpha_s=\mu+\nu, |\mu|<m_s} C_{\mu,\nu} \left| R_{m_s-|\mu|}(D^\mu A_{Q_d}; y, x_0) D^\nu \varphi_d(y) \right| \\
&\quad + \sum_{|\alpha_s|=m_s} \left| \left(D^{\alpha_s} A_s(y) - m_{\overline{Q}_d}(D^{\alpha_s} A_s) \right) \varphi_d(y) \right| \\
&\leqslant C \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} + \sum_{|\alpha_s|=m_s} \left| D^{\alpha_s} A_s(y) - m_{\overline{Q}_d}(D^{\alpha_s} A_s) \right| \\
&\leqslant C \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} + \sum_{|\alpha_s|=m_s} \frac{1}{|\overline{Q}_d|^{1-\beta_s/n}} \int_{\overline{Q}_d} \sup_{y\in 6\sqrt{n}Q_d} \frac{|D^{\alpha_s} A_s(y) - D^{\alpha_s} A_s(x)|}{|y-x|^{\beta_s}} \mathrm{d}x \\
&\leqslant C \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}}, \qquad s = 1,\, 2.
\end{aligned}$$

So, by Lemma 5 we obtain that

$$\|I_1\|_p \leqslant \|A_s^{\varphi_d}\|_\infty \|A_2^{\varphi_d}\|_\infty \|\widetilde{T}_j^0 f_d\|_p \leqslant C \prod_{s=1}^2 \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} 2^{-j\theta} \|f_d\|_p, \tag{3.10}$$

where $\theta = \delta + \delta l/k - \beta > 0$ for $p = 2$, and $\theta = \delta(1/2 - |1/p - 1/2|) + \delta(1/2 - |1/p - 1/2|)l/k - \beta > 0$ for $1 < p < \infty$ $(p \neq 2)$. Similarly,

$$\begin{aligned}
\|I_2\|_p &\leqslant C \|A_1^{\varphi_d}\|_\infty \sum_{0<|\alpha_2|<m_2} \|\widetilde{T}_j^{\alpha_2}(D^{\alpha_2} A_2^{\varphi_d} f_d)\|_p \\
&\leqslant C \|A_1^{\varphi_d}\|_\infty \sum_{0<|\alpha_2|<m_2} 2^{-j\theta} \|D^{\alpha_2} A_2^{\varphi_d}\|_\infty \|f_d\|_p \\
&\leqslant C \prod_{s=1}^2 \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} 2^{-j\theta} \|f_d\|_p; \\
\|I_3\|_p &\leqslant C \|A_1^{\varphi_d}\|_\infty \sum_{|\alpha_2|=m_2} \|\widetilde{T}_j^{\alpha_2}(D^{\alpha_2} A_2^{\varphi_d} f_d)\|_p \\
&\leqslant C \|A_1^{\varphi_d}\|_\infty \sum_{|\alpha_2|=m_2} 2^{-j\theta} \|D^{\alpha_2} A_2^{\varphi_d}\|_\infty \|f_d\|_p \\
&\leqslant C \prod_{s=1}^2 \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} 2^{-j\theta} \|f_d\|_p; \\
\|I_4\|_p &\leqslant C \|A_2^{\varphi_d}\|_\infty \sum_{0<|\alpha_1|<m_1} \|\widetilde{T}_j^{\alpha_1}(D^{\alpha_1} A_1^{\varphi_d} f_d)\|_p \\
&\leqslant C \|A_2^{\varphi_d}\|_\infty \sum_{0<|\alpha_1|<m_1} 2^{-j\theta} \|D^{\alpha_1} A_1^{\varphi_d}\|_\infty \|f_d\|_p \\
&\leqslant C \prod_{s=1}^2 \sum_{|\alpha_s|=m_s} \|D^{\alpha_s} A_s\|_{\dot{\Lambda}_{\beta_s}} 2^{-j\theta} \|f_d\|_p; \\
\|I_5\|_p &\leqslant C \|A_2^{\varphi_d}\|_\infty \sum_{|\alpha_1|=m_1} \|\widetilde{T}_j^{\alpha_1}(D^{\alpha_1} A_1^{\varphi_d} f_d)\|_p \\
&\leqslant C \|A_2^{\varphi_d}\|_\infty \sum_{|\alpha_1|=m_1} 2^{-j\theta} \|D^{\alpha_1} A_1^{\varphi_d}\|_\infty \|f_d\|_p
\end{aligned}$$

$$\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f_d\|_p;$$

$$\begin{aligned}\|I_6\|_p &\leqslant C\sum_{0<|\alpha_1|<m_1}\sum_{0<|\alpha_2|<m_2}\|\widetilde{T}_j^{\alpha_1+\alpha_2}(D^{\alpha_1}A_1^{\varphi_d}D^{\alpha_2}A_2^{\varphi_d}f_d)\|_p\\ &\leqslant C\sum_{0<|\alpha_1|<m_1}\sum_{0<|\alpha_2|<m_2}2^{-j\theta}\|D^{\alpha_1}A_1^{\varphi_d}\|_\infty\|D^{\alpha_2}A_2^{\varphi_d}\|_\infty\|f_d\|_p\\ &\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f_d\|_p;\end{aligned}$$

$$\begin{aligned}\|I_7\|_p &\leqslant C\sum_{0<|\alpha_1|<m_1}\sum_{|\alpha_2|=m_2}\|\widetilde{T}_j^{\alpha_1+\alpha_2}(D^{\alpha_1}A_1^{\varphi_d}D^{\alpha_2}A_2^{\varphi_d}f_d)\|_p\\ &\leqslant C\sum_{0<|\alpha_1|<m_1}\sum_{|\alpha_2|=m_2}2^{-j\theta}\|D^{\alpha_1}A_1^{\varphi_d}\|_\infty\|D^{\alpha_2}A_2^{\varphi_d}\|_\infty\|f_d\|_p\\ &\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f_d\|_p;\end{aligned}$$

$$\begin{aligned}\|I_8\|_p &\leqslant C\sum_{|\alpha_1|=m_1}\sum_{0<|\alpha_2|<m_2}\|\widetilde{T}_j^{\alpha_1+\alpha_2}(D^{\alpha_1}A_1^{\varphi_d}D^{\alpha_2}A_2^{\varphi_d}f_d)\|_p\\ &\leqslant C\sum_{|\alpha_1|=m_1}\sum_{0<|\alpha_2|<m_2}2^{-j\theta}\|D^{\alpha_1}A_1^{\varphi_d}\|_\infty\|D^{\alpha_2}A_2^{\varphi_d}\|_\infty\|f_d\|_p\\ &\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f_d\|_p;\end{aligned}$$

and

$$\begin{aligned}\|I_9\|_p &\leqslant C\sum_{|\alpha_1|=m_1}\sum_{|\alpha_2|=m_2}\|\widetilde{T}_j^{\alpha_1+\alpha_2}(D^{\alpha_1}A_1^{\varphi_d}D^{\alpha_2}A_2^{\varphi_d}f_d)\|_p\\ &\leqslant C\sum_{|\alpha_1|=m_1}\sum_{|\alpha_2|=m_2}2^{-j\theta}\|D^{\alpha_1}A_1^{\varphi_d}\|_\infty\|D^{\alpha_2}A_2^{\varphi_d}\|_\infty\|f_d\|_p\\ &\leqslant C\prod_{s=1}^{2}\sum_{|\alpha_s|=m_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}2^{-j\theta}\|f_d\|_p,\end{aligned}$$

where θ is the same as in (3.10). This completes the proof of Theorem 1 in the case $\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|=1$.

Now we turn to the case $\sum_{|\xi|=k,\,|\eta|=l}|a_{\xi\eta}|\neq 1$. Letting a be as in Theorem 1, we can write $P(x,y)$ as follows

$$P(x,y)=\sum_{|\xi|=k,\,|\eta|=l}\frac{a_{\xi\eta}}{a^{k+l}}(ax)^\xi(ay)^\eta+R_0\Big(\frac{ax}{a},\frac{ay}{a}\Big):=Q(ax,ay).$$

Then

$$\begin{aligned}T_{A_1,A_2}f(x)&=\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}Q(ax,ay)}\frac{\Omega(x-y)}{|x-y|^{n+M}}\prod_{s=1}^{2}R_{m_s+1}(A_s;x,y)f(y)\mathrm{d}y\\ &=\int_{\mathbf{R}^n}\mathrm{e}^{\mathrm{i}Q(ax,y)}\frac{\Omega(ax-y)}{|ax-y|^{n+M}}\prod_{s=1}^{2}R_{m_s+1}\Big(a^{m_s}A_s;\frac{ax}{a},\frac{y}{a}\Big)f\Big(\frac{y}{a}\Big)\mathrm{d}y.\end{aligned}$$

Consequently,

$$T_{A_1,A_2}f(\frac{x}{a}) = \int_{\mathbf{R}^n} \mathrm{e}^{\mathrm{i}Q(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+M}} \prod_{s=1}^{2} R_{m_s+1}(a^{m_s}A_s(a^{-1}\cdot); x, y) f(\frac{y}{a})\mathrm{d}y.$$

Since $\|D^{\alpha_s}(a^{m_s}A_s(a^{-1}\cdot))\|_{\dot{\Lambda}_{\beta_s}} = a^{-\beta_s}\|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}}$ for $|\alpha_s| = m_s$ $(s = 1, 2)$, by the result proved previously, we obtain that

$$\begin{aligned}\|T_{A_1,A_2}f\|_p &\leqslant a^{-\beta}C \prod_{s=1}^{2} \sum_{|\alpha_s|=m_s} \|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}} \|f\|_p \\ &= C(n, a, M, \mathrm{deg}P) \prod_{s=1}^{2} \sum_{|\alpha_s|=m_s} \|D^{\alpha_s}A_s\|_{\dot{\Lambda}_{\beta_s}} \|f\|_p.\end{aligned}$$

Theorem 1 is proved.

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