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Quasi-Topology and Weightable Sets for Complex Generalized Weights

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Abstract In this paper we discuss the relations of the Borel sets, fine Borel sets and quasi-open sets, and the notion of weightable sets for complex generalized weights is introduced.

Key words Complex generalized weights; Choquet property; Fine open sets; Weightable sets
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1 Let $\langle X, T\rangle$ be a Hausdorff topology space. Let Y be a set, H a convex cone composed of non-negative numberical functions on Y, which satisfies the upper directed axiom. Let $\mathscr{F}$ be an extended complex-valued function family such that $\mathscr{R}_e^{\pm}\mathscr{F}$, $\mathscr{I}_m^{\pm}\mathscr{F}$ and $|\mathscr{F}|$ are all included in H. A map $W: 2^X \to \mathscr{F}$ is called a complex p-weight[4,5] on X if $W(\varnothing) = 0$.

In our paper we use the notions and notations in [5, 6, 7]. Here and hereafter, the alphabet K is the positive constant with respect to the notions with "para", and the prefix "para-" will be omitted in relative notions if $K = 1$. Furthermore, in an assertment the term "totally" will be used for a complex weight W in order to mean the assertment is fulfilled for each W_j, $1 \leqslant j \leqslant 4$, where

$$W_1 = W_R^+,\ W_2 = W_R^-,\ W_3 = W_I^+,\ W_4 = W_I^-.$$

2 Assume there is another topology $\tilde{T}$ finer than T in X.

Definition 1 A complex weight W is said to be fine if $W(\tilde{A}) = W(A)$ for any $A \in 2^X$.

Obviously $W_M := \sqrt{W_R^2 + W_I^2}$ is fine if W is; W is fine if and only if each W_j is, $1 \leqslant j \leqslant 4$.

Definition 2 Let W be a complex p-weight. If each fine open set is W-quasi-(strongly W-quasi-) open set, then we say that W has the generalized Choquet (resp. strongly Choquet) property, or W is of generalized Choquet (resp. strongly Choquet) type.

Lemma 1 For a totally increasing complex weight W, the following statements are equivalent:

1) W is of generalized Choquet (resp. strongly Choquet) type;

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2) W_M is of generalized Choquet (resp. strongly Choquet) type;

3) Each W_j is of generalized Choquet (resp. strongly Choquet) type, $1 \leqslant j \leqslant 4$.

Remark The statements 1) and 2) are equivalent provided that W is a complex p-weight.

Theorem 1 Let W be a complex weight of generalized Choquet type.

(i) If W_M is para-increasing, then each fine closed set can be represented a union of a F^e-set and a W-null-set;

(ii) If W_M is sub-strong para-continuous to the right, then each fine Borel set (resp. fine closed set) can be represented the union of a Borel set (resp. F^e-set) and a W-null-set.

Proof (i) Since W is of generalized Choquet type, by Lemma 1, each fine closed set E is W-quasi-closed set, i.e. W_M-quasi-closed set[5]. Then there is a real function $u \in H$ such that for any natural number n there exists a closed set $F_n \subset E$ such that $W_M(E \setminus F_n) \leqslant \frac{1}{n} u$. Write $E_0 = E \setminus \bigcup_{j=1}^{\infty} F_j$, then $E = \bigcup_{j=1}^{\infty} F_j \cup E_0$ and

$$W_M(E_0) = W_M(E \setminus \bigcup_{j=1}^{\infty} F_j) \leqslant K W_M(E \setminus F_n) \leqslant \frac{K}{n} u, \quad n = 1, 2, \cdots,$$

which implies $W(E_0) = 0$. So (i) is proved.

(ii) Since W_M is sub-strongly para-continuous to the right, W_M is para-increasing. Consequently, by (i), each fine closed set can be represented the union of a F^e-set and a W-null-set.

For the Boral algbra $\mathscr{B}$ on X, we prove the family $\mathscr{B}_0 := \{B \cup A \mid B \in \mathscr{B}, W(A) = 0\}$ is a ^{e}-algbra, too. Because the intersection of countabl sets in $\mathscr{B}_0$ belongs obviously to $\mathscr{B}_0$, it is sufficient to prove that for a Boral set B and a W-null-set E, the complement $\complement(B \cup E)$ of $B \cup E$ can be represented the union of a Borel set and a W-null-set. In fact, we have $W_M(E) = 0$. Since W_M is sub-strong para-continuity to the right, then there is a real function $u \in H$ such that for any natural number n there exists an open set $G_n \supset E$ such that

$$W_M(G_n) \leqslant K W_M(E) + \frac{1}{n} u = \frac{1}{n} u. \tag{1}$$

Let $G = \bigcap_{n=1}^{\infty} G_n$. we have $\complement(B \cup E) = (G \setminus E \cup B) \cup \complement(B \cup G)$. Since $W_M(G) \leqslant K W_M(G_n) \leqslant \frac{K}{n} u$ for any n, then $W_M(G) = 0$, which implies $W(G \setminus E \cup B) = 0$. Therefore $\mathscr{B}_0$ is a ^{e}-algbra is proved.

By (i), each fine closed set belongs to $\mathscr{B}_0$. We have $\mathscr{B}^* \subset \mathscr{B}_0$, because the fine Borel algebra $\mathscr{B}^*$ is a mininun σ-ring which contains all fine closed sets. (ii) is proved. Q. E. D.

Remark If we replace "W_M is" by "W is totally" in (i) and (ii), the results still hold.

Theorem 2 Let W be a complex weight. If W_M is para-increasing and finitely (resp. countably) para-subadditive, then the union of finite W-quasi-(resp. countable strongly W-quasi-) open sets is W-quasi-open set.

Proof The following is only a proof to the countable case. Let $\{A_n\}$ be a sequence of strong W-quasi-open sets (i.e. W_M-quasi-open sets[5]). Then for any positive integer n, there exists a

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bounded function $u_n \in$ H (without loss of generality, assume $u_n \leqslant 1$) such that for any X > 0 there exists an open set $G_n \supset A_n$ satisfying $W_M(G_n \setminus A_n) \leqslant \frac{\mathrm{X} u_n}{\mathrm{K}^2 2^n}$. Since $\bigcup_{n=1}^{\infty} G_n \setminus \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} (G_n \setminus A_n)$, then

$$W_M\left(\bigcup_{n=1}^{\infty} G_n \setminus \bigcup_{n=1}^{\infty} A_n\right) \leqslant K\, W_M\left(\bigcup_{n=1}^{\infty} (G_n \setminus A_n)\right) \leqslant K \sum_{n=1}^{\infty} W_M(G_n \setminus A_n) \leqslant \mathrm{X} \sum_{n=1}^{\infty} \frac{u_n}{2^n}.$$

By taking $u = \sup_n u_n \in$ H, the proof is hereby completed. Q. E. D.

Corollary Let W be a complex weight totally para-increasing and totally finitely (resp. countably) subadditive. Then the union of finite (resp. countable) W-quasi-open sets is W-quasi-open set.

Theorem 3 Let W be a fine complex weight. If W_M is para-increasing, then each W-quasi-open set is the union of a fine open set and a W-null-set.

Proof Let G is a W-quasi-open set. Denote by $\mathring{G}$ the fine inner part of G. It is sufficient to prove $W(G \setminus \mathring{G}) = 0$. By Theorem 2.1 in [6], the characteristic funcion i_G of G is W-quasi-lower semicontinuous. Because W is fine, by Theore 2.1 in [7], i_G is fine lower semicontinuous W-quasi-everywhere, i.e. there exists an set E with $W(E) = 0$ such that $\mathrm{i}_G|C_E$ is fine semi-lower continuous. Obviously $\mathrm{i}_G(x) = 1$ for any $x \in G \setminus \mathring{G}$. Since $x \overline{\in} \mathring{G}$, the lower limit of i_G at x is 0. Consequntly i_G is not fine semi-lower continuous at $x \in G \setminus \mathring{G}$, which implies $G \setminus \mathring{G} \subset E$. Therefore, $W(G \setminus \mathring{G}) = 0$. Q. E. D.

Corollary Let W be a fine complex weight with W_M continuous (or sub-srongly continuous) to the right. Then each W-quasi-open set is the union of a fine open set and a W-null-set.

Theorem 4 Let W be a fine complex weight of generalized Choquet type.

(i) If W_M is increasing, then for any set $E \subset X$ there is a real function $u \in$ H such that for any number X > 0, there exists two sets E_1, E_2 such that

$$W_M(\tilde{E}_1) \leqslant W_M(E),\ W_M(E_2) \leqslant \mathrm{X} u, \text{ and } E = E_1 \cup E_2. \tag{2}$$

(ii) If W is totally increasing, then for any set $E \subset X$ the relation (2) is fulfiled also, and there is a real function $u \in$ H such that for any number X > 0, there exists two sets E_1, E_2 such that

$$W_j(\tilde{E}_1) \leqslant W_j(E),\ W_j(E_2) \leqslant \mathrm{X} u\ (1 \leqslant j \leqslant 4), \text{ and } E = E_1 \cup E_2.$$

Proof Here we give only the proof of (ii). First, by Lemma 1, W_M and each W_j ($1 \leqslant j \leqslant 4$) are of generalized Choquet type. Because each W_j is of generalized Choquet type, the fine closure $\tilde{E}$ of E is W_j-quasi-closed set. Then there exists a real function $u_j \in$ H such that for any number X > 0, there exists a closed set $F_j \subset \tilde{E}$ satisfying the condition $W_j(\tilde{E} \setminus F_j) \leqslant \mathrm{X} u_j$. Since W_j be a fine and increasing, we have $W_j(F_j) \leqslant W_j(\tilde{E}) = W_j(E)$. Write $E_1 = E \cap \left(\bigcup_{i=1}^{4} F_i\right)$, $E_2 = E \setminus \bigcup_{i=1}^{4} F_i$ and $u = \max\{u_1, u_2, u_3, u_4\}$, then $E = E_1 \cup E_2$, $u \in$ H and

$$W_j(\tilde{E}_1) = W_j(E_1) \leqslant W_j\left(\bigcup_{i=1}^{4} F_i\right) \leqslant W_j(\tilde{E}) = W_j(E),$$

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$$W_j(E_2) = W_j(E \setminus \bigcup_{i=1}^{4} F_i) \leqslant W_j(\tilde{E} \setminus F_j) \leqslant X_{uj} \leqslant X_u.$$ Q. E. D.

Definition 3 A relative compat sub-set E of X is called W-almost open-closed set if its boundary ∂E is W-null-set, i.e. $W(\partial E) = W(\bar{E} \setminus \mathring{E}) = 0$.

Theorem 5 For a complex weight W, each W-almost open-closed set is either W-quasi-open set or W-quasi-closed set provied that W_M is sub-strong para-continuous to the right and para-sub-additive.

Proof First, W_M is para-increasing because of sub-strong para-continuity to the right.

Second, we prove that the result is hold for each W-null-set. Since W_M is sub-strong para-continuous to the right, there exists a real function $u \in H$ such that for any number $X > 0$, there exists a open set $G \supset A$ satisfying the condition $W_M(G) \leqslant K W_M(A) + \frac{X_u}{K} = \frac{X_u}{K}$. Consequently

$$W_M(G \setminus A) \leqslant K W_M(G) \leqslant X_u.$$

So A is W_M-quasi-open, i.e. W-quasi-open[5]. And for any closed set $F \subset A$, $W(A) = 0$ implies $W(A \setminus F) = 0$. Therefore A is W_M-quasi-closed, i.e. W-quasi-closed.

Finally, let E be a W-almost open-closed set. Because $E = \mathring{E} \cup (E \cap \partial E)$ and $W(E \cap \partial E) = 0$, E is W-quasi-open. In the other hand, $E = \bar{E} \cap C(\partial E \setminus E)$ and $W(\partial E \setminus E) = 0$, which implies $C(\partial E \setminus E)$ is either W-quasi-open or W-quasi-closed. The rerult follows from Theorem 2.

Q. E. D.

Corollary Let W be a complex weight. If W_M is para-sub-additive and sub-strong para-continuous to the right para-sub-additive, then the characteristic function i_E of a W-almost open-closed set E is W-quasi-continuous.

Proof It follows from Theorem 2.1 in [6] and Theorem 5. Q. E. D.

3. A complex weight W is said to be C-monotone if $W(\bigcup_{j=1}^{\infty} F_j) = \lim_{j \to \infty} W(F_j)$ for any monotonically increasing sequence $\{F_j\}_{j=1}^{\infty}$ of the relative compact subsets of X.

Let P be a non-negative increasing weight. For a subset A of X, set

$$P^*(A) = \inf\{P(G) \mid G \text{ is the open set and } G \supset A\},$$
$$P_*(A) = \sup\{P(F) \mid F \text{ is the compact subset of } A\}.$$

For a totally increasing complex weight W, write

$$W^*(A) = W_1^*(A) - W_2^*(A) + i[W_3^*(A) - W_4^*(A)];$$
$$W_*(A) = (W_1)_*(A) - (W_2)_*(A) + i[(W_3)_*(A) - (W_4)_*(A)].$$

We say W is continuous (resp. quasi-continuous) to the right if $W(A) = W^*(A)$ for any $A \in 2^X$ (resp. for any compact subset A of X). Evidently, if W is continuous to the right, then W is totally increasing.

Definition 4 A set $A \subset X$ is said to be W-weightable if $W_*(A) = W(A)$.

Lemma 5 For a complex weight W, (i) W is C-monotone if and only if each W_j is, $1 \leqslant j \leqslant 4$; (ii) W_M is C-monotone if W is; (iii)[6] $(W^*)_M(A) = (W_M)^*(A)$, $\forall A \in 2^X$; therefore W_M is (quasi-) continuous to the right if W is.

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Lemma 6 $(W_*)_M = (W_M)_*$ for a totally increasing complex weight W.

Proof Let $A \in 2^X$. Obviously, $(W_*)_M \geqslant (W_M)_*$. For the inverse inequility, without loss of generality, assume $(W_M)_*(A) < \infty$, then there are two sequences $\{F_j^1\}_{j=1}^{\infty}$ and $\{F_j^2\}_{j=1}^{\infty}$ of compact subsets of X such that $F_j^1, F_j^2 \subset A$ and $\lim_{j\to\infty} W_R(F_j^1) = (W_R)_*(A)$, $\lim_{j\to\infty} W_I(F_j^2) = (W_I)_*(A)$. Let $F_j = F_j^1 \cup F_j^2$, then we have $\lim_{j\to\infty} W_R(F_j) = (W_R)_*(A)$ and $\lim_{j\to\infty} W_I(F_j) = (W_I)_*(A)$. Therefore

$$(W_*)_M(A) = \sqrt{[(W_R)_*(A)]^2 + [(W_I)_*(A)]^2}$$

$$= \lim_{j\to\infty} \sqrt{[W_R(F_j)]^2 + [W_I(F_j)]^2} \leqslant (W_M)_*(A). \qquad \text{Q. E. D.}$$

Corollary A W-weightable set is W_M-weightable.

Now the results in [10] could be used to discuss the weightable sets for complex weights, such as:

Theorem 6 Let X be a locally compact T_2-space. Let W be a complex weight on X, which is totally increasing, C-monotone and quasi-continuous to the right. Then (i) each $K^{\sigma\delta}$-set of X is W-weightable; (ii) each K-analytic set[9] of X is W-weightable.

Theorem 7 Let X be a locally compact T_2-space with countable base. Let W be a complex weight on X, which is totally increasing, C-monotone and quasi-continuous to the right. We have (i) each Borel set of X is W-weightable; (ii) if W is also totally subadditive, then all the W-quasi-open sets and W-quasi-closed sets are W-weightable; (iii) if W is also fine[7] and totally subadditive, then the fine closure of each W-quasi-closed set is W-weightable, and so is the fine inner part of each W-quasi-open set.

Theorem 8 Let X be a locally compact T_2-space with countable base, W a complex weight on X. If W_M is increasing, C-monotone, quasi-continuous to the right, subadditive and of Choquet type,[7] then (i) all the fine open sets and fine closed sets are W_M-weightable; (ii) each fine Borel set of X is W_M-weightable provided that W_M is also sub-strongly para-continuous to the right.[7]

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(下转第 28页)

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含各阶导数的非线性弹性梁方程的一个存在定理

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摘 要 通过选择适当的 Banach空间并利用 Leray-Schauder非线性抉择对于含各阶导数的非线性弹性梁方程

$$\begin{cases} u^{(4)}(t)= f(t, u(t), u'(t), u''(t), u'''(t)), & 0\leq t\leq 1, \\ u(0)= u'(1)= u''(0)= u'''(1)= 0 \end{cases}$$

建立了一个解的存在定理. 在材料力学中,该方程描述了一端简单支撑,另一端被滑动夹子夹住的弹性梁的形变. 这个存在定理说明只要非线性项满足某种线性增长条件该方程至少有一个解.

关键词 非线性弹性梁方程; 边值问题; 存在性

(上接第 23页)

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摘 要 本文讨论了 Hausdorff空间中 Borel集、细 Borel集和拟开集之间的关系; 引入复广义可权集的概念并讨论其性质.

关键词 复广义权; Choquet性质; 细开集; 可权集

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