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On KL-convergent graphs

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Abstract The line-clique graph, $KL(G)$, of a graph G is the intersection graph of the cliques of the line graph $L(G)$ of G. For a natural number n, a graph G is n-KL-convergent if the nth iterated-line-clique graph $KL^n(G)$ is isomorphic to a fixed graph G'. A graph G is KL-convergent if there is a natural number n such that G is n-KL-convergent. Otherwise G is divergent.

In this paper, KL-convergent graphs and divergent graphs are characterized. A method for constructing any KL-convergent graph from a graph triangle free is provided. We also discuss the KL-convergent index of KL-convergent graphs.

Key words line-clique graph; KL-convergent; KL-convergent index

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1 Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. The *line graph* $L(G)$ of G, as usual, is defined to be the graph whose vertex set is $E(G)$, and two vertices are adjacent provided their corresponding edges are adjacent in G. A maximal complete subgraph of a graph G is a *clique* and the *clique graph* $K(G)$ of G is the intersection graph of the clique of G. Following them, we define the *line-clique graph*, $KL(G)$, which is the clique graph of the line graph of G.

The convergency of clique graphs was introduced and studied in [1, 2, 3]. The nth iterated-clique graph of a graph G is defined by $K^n(G) := K(K^{n-1}(G))$ and G is n-convergent if $K^n(G)$ is isomorphic to K_1 (the one vertex graph) in [1]. In this paper, we discuss the convergency of line-clique graphs.

Definition 1 The nth *line-clique graph* of a graph G is analogously defined by $KL^n(G) := KL(KL^{n-1}(G))$.

Definition 2 A graph G is *n-KL-convergent* to a graph G' if for any natural number $m(\geq n\geq 1)$, $KL^m(G)$ is isomorphic to G'.

Definition 3 A graph G is said to be *KL-convergent* or a *KL-convergent* graph if it is n-KL-convergent to a graph G' for some natural number $n(\geq 1)$. Otherwise we call G *divergent*.

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Definition 4 Suppose G is KL-convergent. The smallest n such that G is n-KL-convergent is called the *KL-convergent index* of G, denoted by $n(G)$.

Some terminology and notations can be found in [5, 6].

2 The convergency of line-clique graphs

For a graph G, let G^* denote the graph obtained from G by the following steps:

(1) for each triangle T in G, inserting a new vertex v_T,

(2) removing all vertices of degree 1 in G and all the vertices of degree 2 contained in triangles,

(3) for each triangle T in G, connecting v_T with the vertices of T which are not removed by (2); for triangles T and T', connecting v_T with $v_{T'}$ if T and T' have two vertices in common.

The process is illustrated as follows.

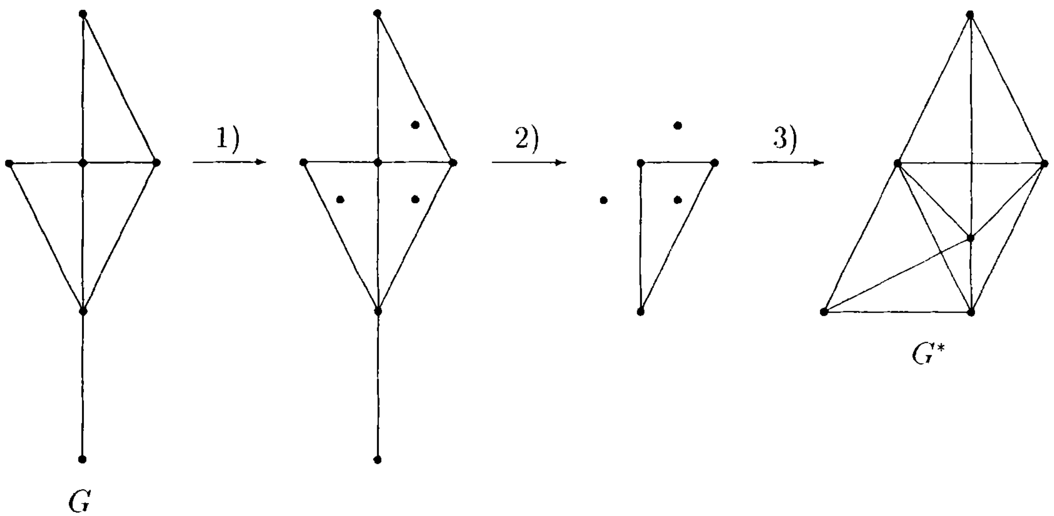

Fig. 1

In order to characterize the convergency of line-clique graphs, the following lemmas are necessary.

Lemma 2.1[4] $KL(G) = G^*$. ◇

Lemma 2.2[4] $KL(G) = G$ if one of the following holds:

(a) G has no vertices of degree 1 and no triangles;

(b) G has no vertices of degree 1. Each triangle of G has exactly a vertex of degree 2. And each pair of triangles has at most one vertex in common. ◇

Theorem 2.1 Let G be a connected graph and let $k(G)$ be the order of the maximum clique in G. Then G is KL-convergent if and only if the following three conditions hold:

(1) $k(G) \leq 3$,

(2) each triangle of G has at least one vertex of degree 2,

(3) G has no a pair of triangles having two vertices in common.

Proof As shown in Lemma 2.1, all vertices of degree 1 of a graph G are removed in its line-clique graph $KL(G)$. Thus there is a natural number n such that $KL^n(G)$ has no vertex of degree 1.

We prove the necessity of the conditions by contradiction.

Suppose that $k(G) \geq 4$. Then there is a clique K_m ($m \geq 4$) in G. By Lemma 2.1, all vertices and edges of K_m aren't removed in $KL(G)$. Furthermore, for each triangle of K_m, a new vertex and three edges are added to $KL(G)$. So in $KL(G)$, there must be triangles with no vertex of degree 2. By Lemma 2.1, it is impossible to get a natural number $n(\geq 1)$ such that $KL^n(G)$ is isomorphic to a graph G'. Hence G is divergent, a contradiction.

Suppose that the condition (2) is not true, that is, there is a triangle T with no vertex of degree 2. Then T becomes K_4 in $KL(G)$. By the analogous discussion above, G is divergent.

Suppose that the condition (3) doesn't hold. Then G_1 (showed in Fig. 2) is a subgraph of G, and G_1 becomes K_4 in $KL(G)$. So G is divergent, a contradiction.

Now we proceed to prove the sufficiency. The vertices of degree 1 in G will be deleted in $KL(G)$. For those triangles having exactly one vertex of degree 2, they don't change in $KL(G)$. In other words, they still have exactly one vertex of degree 2. For those triangles having two vertices of degree 2, in $KL(G)$ they become K_2 with one vertex of degree 1. In words, they will be deleted finally. Thus there must be a natural number $n(\geq 1)$ such that $KL^n(G)$ satisfies the conditions of Lemma 2.2. Hence G is KL-convergent.

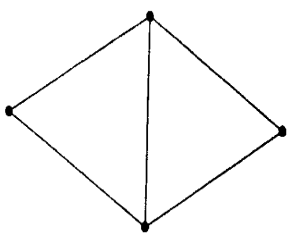

Fig. 2

We complete the proof. ◇

3 Construction and decomposition of KL-convergent graphs

Decomposition Let G be a KL-convergent graph. From Theorem 2.1, each triangle of G has at least one vertex of degree 2. Thus we can get a subgraph H of G by deleting all the vertices of degree 2 lying on triangles in G. Notice that H is a graph free of triangles. By Theorem 2.1, H is still a KL-convergent graph.

In reverse, we can construct any KL-convergent graph G from a graph free of triangles as follows.

Construction Let H be a graph free of triangles and let $E_1 \subseteq E(H)$, $V_1 \subseteq V(H)$. For each edge $e = uv$ in E_1, add a path of length 2 connecting u and v. For every vertex v in V_1, add a triangle T so that one vertex of T is just v. We use G to denote the resulted graph. By Theorem 2.

1, G is a KL-convergent graph.

4 KL-convergent index of KL-convergent graphs

In this section, we discuss the KL-convergent index $n(G)$ of a KL-convergent graph G by distinguishing three cases.

Look at the blocks of G now. They can be divided into the following three classes. One is K_2, called *T_1-block*. Another is a triangle, called *T_2-block*. The third, called *T_3-block*, is such block that is neither T_1-block nor T_2-block and confirms to the three conditions Theorem 2.1 mentioned. During the process of getting $KL^{n(G)}(G)$, T_1-block can be fixed or be deleted; T_2-block can be fixed or become K_2 and then be deleted; T_3-block is fixed from beginning. So we say T_1-block can contribute 1 to $n(G)$, T_2-block can contribute 2 to $n(G)$ and T_3-block contributes 0 to $n(G)$.

Case 1 If G has no T_3-block, then $n(G)= [l/2]$ where l is the length of the longest path in G. Furthermore, G is KL-convergent to K_1. Indeed by the Lemma 2.1, T_1-block can contribute 1 to $n(G)$ and T_2-block can contribute 2 to $n(G)$. Notice that a path of length 2 as a subgraph can contribute 2 to $n(G)$, too. So if G'' is the graph obtained from G by replacing T_2-block with a path of length 2, then $n(G)= n(G'')$. It is clear that G'' is a tree. Hence $n(G'')= [d(G'')/2]$ where $d(G'')$ is the diameter of G''. On the other hand, $d(G'')= l$. Hence $n(G)= [l/2]$ and $KL^{n(G)}(G)= K_1$.

For G with blocks $\{B_i\}$ and cutvertices $\{c_j\}$, the *block-cutvertex graph*, denoted by $bc(G)$ [7], is defined as the graph having vertex set $\{B_i\}\cup\{c_j\}$, with two vertices adjacent if one corresponds to a block B_i and the other to a cut vertex c_j and c_j is in B_i. Thus $bc(G)$ is a tree in which the distance between any two endvertices is even. For the sake of convenience, if a block is T_i-block, then the vertex in $bc(G)$ corresponding to it will be marked with b_i, $i= 1, 2, 3$. There is no danger of confusion if the cutvertices of G are all denoted by v. An example is showed in Figure 3.

Case 2 If all the endvertices of $bc(G)$ are marked with b_3, then $n(G)= 0$, that is, G is KL-convergent to itself. Suppose P is a path whose endvertices are both marked with b_3 in $bc(G)$. Then all the blocks which correspond to the vertices on P contribute 0 to $n(G)$. We call such path a *b_3-path*. (In Fig. 3, the path marked with thick lines is a b_3-path.) If all the endvertices of $bc(G)$ are marked with b_3, then every vertex of $bc(G)$ is on some b_3-path. Thus all blocks of G don't change in $KL(G)$, that is, $KL(G)= G$. Then we got the result.

Case 3 Suppose G has at least a T_3-block and at least an endvertex of $bc(G)$ is not marked with b_3.

Let $bc_1(G)$ be the subgraph of $bc(G)$ by deleting all b_3-paths. Assume that there are t components in $bc_1(G)$. Clearly, the vertices of these t components aren't marked with b_3. We use G' to denote the subgraph of G which corresponds to the subgraph of $bc(G)$ induced by all the b_3-paths. $G[E(G)\setminus E(G')]$ is divided into t parts G_1, G_2, $\cdots$, G_t (they are different from

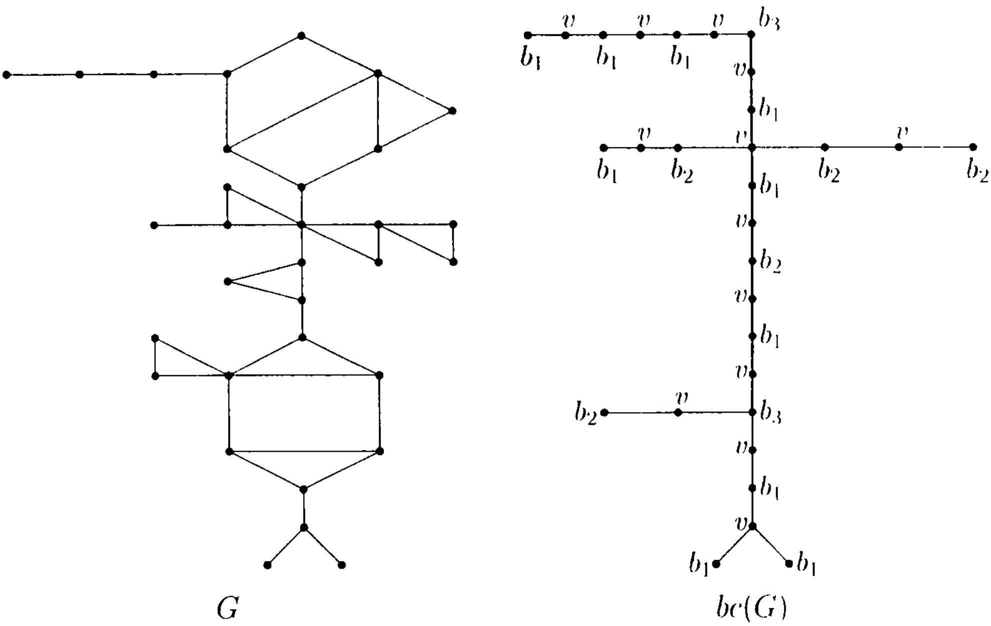

Fig. 3

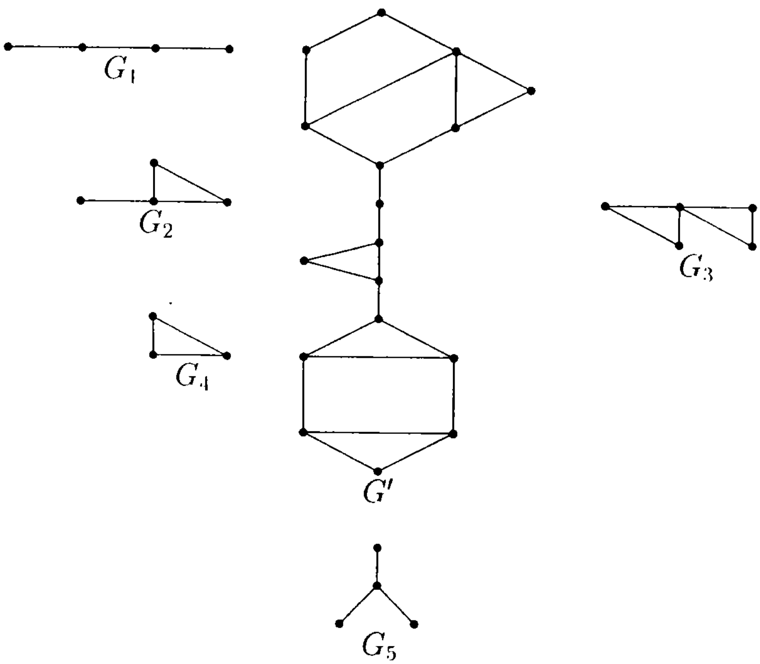

Fig. 4

components). Each of them corresponds exactly to a component of $bc^1(G)$. Thus G_i contains no T_3-block where $i\in\{1,2,\cdots,t\}$. With the replacement similar to Case 1 done to every G_i, the following result can be got quickly. Then $n(G)=\max_i\{l_i \mid l_i$ is the length of the longest path in $G_i\}$ and G is KL-convergent to G'.

We take G showed in Fig. 3 for an example. G' is showed in Fig. 4. There are 5 parts in $G[E(G)\setminus E(G')]$, say $G_1, G_2,\cdots, G_5$ also showed in Fig. 4. Hence $n(G)=4$ and G is KL-convergent to G'.

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线团-收敛图

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摘 要 一个图的线团图就是这个图的线图的团图. 对于自然数 n,一个图被称为 n-线团-收敛的,如果它的 n 次线团图同构于一个固定的图. 否则称之为发散的.

本文刻画了线团-收敛图与发散图,给出一个线团-收敛图的构造方法. 并且,讨论了线团-收敛图的线团-收敛指数.

关键词 线团图; 线团-收敛; 线团-收敛指数