文章编号: 1009-1327 (2001) 03-0208-23

Survey of Recent Developments of Random Metric Theory and Its Applications in China (Ⅱ)

GUO Tie-xin

Department of Mathematics, Xiamen University, Xiamen Fujian 361005, China

Abstract: The central purpose of this paper is to present a complete account of the principal results and ideas currently available in the course of the development of random metric theory and its applications in China. This paper is divided into ten sections. Section 1 is devoted to a brief introduction of the background of our work—the theories of probabilistic metric spaces and random metric spaces; Section 2 to some preliminaries from random functional analysis and abstract space-valued measurable functions. Section 3 is devoted to a survey of the global relationship between random functional analysis and original random metric theory (called F-random metric theory in this paper): the principal results are smoothly putting random elements, and hence also random operators into the basic frameworks of F-random metric theory by reasonably and naturally constructing F-random metric and F-random norm on the spaces generated by random elements; based on these constructs, we further put together random functional analysis and F-random metric theory to directly lead to a new approach to random functional analysis—the space-randomized approach; besides these, this section also gives, for the first time, the basics of the theory of F-random conjugate spaces (all the results in this section are due to the author). In section 4, based on a new version of random metric theory—E-random metric theory recently presented by the author, we give the basics of the previously developed theory of E-random conjugate spaces (this section mainly consists of the author's work, at the same time Zhu Linhu's important work on random linear functionals on E-norm spaces is in particular mentioned). The following two sections are concerned with the most substantial and deepest parts of the theory of E-random conjugate spaces: Section 5 is devoted to representation theorems of E-random conjugate spaces of several classes of E-random normed modules (this section consists of the author's work, the joint work of the author with YOU Zhao-yong and LIN Xi, and the joint work of LIU Qing-rong with GONG Fu-zhou); Section 6 to characterizations of E-random reflexive spaces (this section consists of the author's work and the joint work of the author with YOU Zhao-yong and others). Section 7 gives the basics of the theory of E-random seminormed modules together with E-random dualities (this section mainly consists of the author's work). Sections 8 and 9 are devoted to a brief elucidation of the relations of random metric theory to functional analysis and the theory of probabilistic metric spaces. Section 10 concludes this paper with a further analysis of the space-randomized approach to random functional analysis.

Key words: the theory of probabilistic metric spaces; random metric theory; random functional analysis; the space-randomized approach to random functional analysis; random normed modules; random inner product modules; random conjugate spaces; random dualities

CLC number: O177.3+9

Received date: 2001-03-16

Foundation item: the National Natural Science Foundation of China (No. 10071063); the Natural Science Foundation of Fujian Province of China (F99027)

5 Representation Theorems for E-random Conjugate Spaces of Several Classes of E-RN Modules

In this section, $(S, \mathscr{X})$ always denotes a complete E-RIP module over K with base $(\Omega, \mathscr{A}, \mu)$, $(S, \tilde{\mathscr{X}})$ the complete E-RN module derived from $(S, \mathscr{X})$; $(B, \|\cdot\|)$ is a normed space over K, the E-RN modules $L(\mu, B)$, $L(\mu, B')$ and $L(\mu, B', w^*)$ are the same as in Example 4.2 and Example 4.3, respectively.

Two elements p and q in $(S, \mathscr{X})$ is called orthogonal if $X_{p,q} = 0$; for a subset M of S, $M^{\perp} = \{q \in S \mid X_{p,q} = 0$ for all p in $M\}$ is called the orthogonal complement of M.

Proposition 5.1[48] Let M be a closed subspace of $(S, \mathscr{X})$. Then $S = M \oplus M^{\perp}$ iff M is a submodule.

Proof If M is a closed submodule, then $\{\tilde{X}_{p-q} \mid q$ in $M\}$ is dually directed, where p is a fixed element in S. In fact, for any two elements q_1, q_2 in M, let $A = [\tilde{X}_{p-q_1} \leq \tilde{X}_{p-q_2}]$, then it is easy to see $\tilde{X}_{p-q_1} \wedge \tilde{X}_{p-q_2} = \tilde{X}_{p-q_3}$, where $q_3 = I_A \cdot q_1 + I_{A^c} \cdot q_2$, which is clearly in M. Thus it follows immediately from Proposition 4.1 that there exists a sequence $\{q_n\}$ in M such that $\{\tilde{X}_{p-q_n} : n \in N\} \searrow \wedge \{\tilde{X}_{p-q} \mid q \in M\}$. Similar to the classical case of Hilbert spaces, one can easily prove $\{q_n : n \in N\}$ is a Cauchy sequence, hence convergent to some unique p_0 in M such that $\tilde{X}_{p-p_0} = \wedge\{\tilde{X}_{p-q} \mid q \in M\}$, which is equivalent to the fact that $p - p_0$ is in $M^{\perp}$, see, e.g., [49, 39, 40]. Thus $S = M \oplus M^{\perp}$.

If $S = M \oplus M^{\perp}$, one can also easily see M must be a submodule.

This completes the proof.

Theorem 5.1[45] f is an a.e. bounded E-random linear functional on $(S, \mathscr{X})$ iff there exists a unique element q in S such that $f(p) = X_{p,q}$ for all p in S, and at this time $\tilde{X}^*_f = \tilde{X}_q$.

Proof We only need to prove sufficiency. We can, without loss of generality, assume $(\Omega, \mathscr{A}, \mu)$ is a probability space.

Denote $\{f(p) \mid p \in S\}$ by Ranf, and $\{p \in S \mid f(p) = 0\}$ by Ker(f). Then Ker(f) is a closed submodule of S, and it is easy to see $\{|f(p)| \mid p \in S\}$ is directed. Denote $\vee\{|f(p)| \mid p \in S\}$ by ξ, then ξ is in general an element in $\bar{L}(\mu, R)$ such that $\xi \geq 0$, and thus by Proposition 4.1 there exists a sequence $\{p_n\}$ in S such that $\{|f(p_n)|\} \uparrow \xi$.

Set $B = [\xi > 0]$ and $B_i = [|f(p_i)| > 0]$ for all i in N, then $\{I_{B_i} \mid i \in N\} \uparrow I_B$. Let $B_0 = \varnothing$ and $A_i = B_i \setminus B_{i-1}$ for all $i \in N$, then $B = \sum_{i=1}^{\infty} A_i$. Put $q_i = f(p_i))^{-1} \cdot p_i$, then $f(q_i) = I_{A_i}$ for all i in N.

If $\mu(B) = 0$, then $q = \theta$ is the desired; if $\mu(B) > 0$, then Ran$f \neq \{0\}$, for the case we can, without loss of generality, assume $\mu(A_i) > 0$ for all $i \in N$. By Proposition 5.1, for each i in N there exists a unique $\tilde{q}_i$ in Ker$(f)^{\perp}$ such that $q_i - \tilde{q}_i \in$ Ker(f). Thus $f(\tilde{q}_i) = f(q_i) = I_{A_i}$, and hence $f(I_{A_i} \cdot p - I_{A_i} f(p) \cdot \tilde{q}_i) = I_{A_i} \cdot f(p) - I_{A_i} \cdot f(p) = 0$ (note: f is a continuous module homomorphism by Theorem 4.1) for all i in N and all p in S. Then we have $\mathscr{X}(I_{A_i} \cdot p - I_{A_i} \cdot f(p) \cdot \tilde{q}_i, \tilde{q}_i) = 0$, i.e., $I_{A_i} \cdot f(p) \cdot \tilde{X}^2_{\tilde{q}_i} = I_{A_i} \cdot X_{p, \tilde{q}_i}$ for all i in N

and all p in S.

Since $|f(p)| \leq \tilde{X}_f^* \cdot \tilde{X}_p$ for all p in S, then $I_{A_i} = f(q_i) = f(\tilde{q}_i) \leq X_f^* \cdot \tilde{X}_{\tilde{q}_i}$ for all i in N. Thus $A_i \subset [\tilde{X}_{\tilde{q}_i} > 0]$, further $I_{A_i} \cdot f(p) = \mathscr{X}(p, q_i^*)$ for all p in S, where $q_i^* = I_{A_i} \cdot (\tilde{X}_{\tilde{q}_i}^2)^{-1} \cdot \tilde{q}_i$ for all i in N.

Put $q_n^0 = \sum_{i=1}^{n} q_i^*$ for all n in N, then $\{q_n^0 \mid n \in N\}$ is a Cauchy sequence since $\mu(B) = \sum_{i=1}^{\infty} \mu(A_i) < +\infty$, hence convergent to some unique element q in S. Notice $I_B \cdot f(p) = f(p)$ for all p in S and $\{I_{B_n}\} \nearrow I_B$, then $f(p) = I_B \cdot f(p) = \lim_{n\to\infty}[I_{B_n} \cdot f(p)] = \lim_{n\to\infty} \sum_{i=1}^{n} (I_{A_i} \cdot f(p)) = \lim_{n\to\infty} \mathscr{X}(p, q_n^0) = X_{p,q}$ by the obvious continuity of $\mathscr{X}$, for all p in S.

It is clear that $\tilde{X}_f^* \leq \tilde{X}_q$ since $|f(p)| = |X_{p,q}| \leq \tilde{X}_p \cdot \tilde{X}_q$ for all p in S. As to $\tilde{X}_q \leq \tilde{X}_f^*$, we proceed as follows.

Let $A = [\tilde{X}_q > 0]$, then from $\tilde{X}_f^* \cdot \tilde{X}_q \geq |f(q)| = \tilde{X}_q^2$ we have $I_A \cdot \tilde{X}_f^* \geq I_A \cdot \tilde{X}_q$, while it is clear that $I_{A^c} \cdot \tilde{X}_f^* \geq 0 = I_{A^c} \cdot \tilde{X}_q$, thus $\tilde{X}_f^* = I_A \cdot \tilde{X}_f^* + I_{A^c} \cdot \tilde{X}_f \geq I_A \cdot \tilde{X}_q + I_{A^c} \cdot \tilde{X}_q = \tilde{X}_q$.

This completes the proof.

Remark 5.1 Riesz representation theorem in complete E-RIP spaces was first studied in [40] where the main result amounts to our Theorem 5.1 except a serious restriction that Ran f is asked to be closed. Theorem 5.1 shows every complete E-RIP module is random self-conjugate, which considerably improves Lax-Milgram theorem [40] and spectral decomposition theorems [66] in complete E-RIP modules by removing the superfluous assumption of random self-conjugateness. As pointed out in [40, 66], these improved results are, without doubt, a powerful tool for random equations and random self-adjoint linear operators on Hilbert spaces.

Definition 5.1[47] Let $(S^1, \mathscr{X}^1)$ and $(S^2, \mathscr{X}^2)$ be two E-RN modules over K with base $(\Omega, \mathscr{A}, \mu)$. Then they are called isometrically isomorphic if there exists a module isomorphism T from S^1 onto S^2 such that $X_{Tp}^2 = X_p^1$ for all p in S^1, briefly $S^1 \cong S^2$ under T.

Proposition 5.2[47] Define $T: L(\mu, B', w^*) \to (L(\mu, B))^*$ (namely the E-random conjugate space of $L(\mu, B)$) as follows: for each q in $L(\mu, B', w^*)$, T_q (denoting $T(q)$): $L(\mu, B) \to L(\mu, K)$ by $T_q(p) = \langle q, p \rangle$ for all p in $L(\mu, B)$. Then $L(\mu, B', w^*) \cong (L(\mu, B))^*$ under T.

Proof See Remark 4.7 for the natural pairing $\langle q, p \rangle$, we only need to prove T surjective.

In fact, let f be a given element in $(L(\mu, B))^*$, then the restriction of f to B, still denoted by f, is a linear operator from B to $L(\mu, K)$ such that $|f(b)| \leq X_f^* \cdot \|b\|$ for all b in B. By Lemma 4.1, there exists q in $L(\mu, B', w^*)$ such that $f(b) = \langle q, b \rangle$ for all b in B, since f and T_q are both continuous module homomorphisms from $L(\mu, B)$ to $L(\mu, K)$ by Theorem 4.1, and since all simple elements are dense in $L(\mu, B)$, $f(p) = T_q(p)$ for all

p in $L(\mu,B)$, namely $f = T_q$. $X_q = X_f^*$ follows immediately from Theorem 4.1 and the definition of the E-random norm on $L(\mu,B',w^*)$.

This completes the proof.

Remark 5.2 A less elegant form of Proposition 5.2 occurred independently in [37, 42] where the terminology "random variables" were used, whereas the probability space in question was not asked to be complete.

Theorem 5.2[47] Define $T: L(\mu,B') \to (L(\mu,B))^*$ as follows: for each q in $L(\mu,B')$, T_q(denoting $T(q)$) $: L(\mu,B) \to L(\mu,K)$ by $T_q(p) = \langle q,p \rangle$ for all p in $L(\mu,B)$. Then $L(\mu,B') \cong (L(\mu,B))^*$ under T iff B' has the Radon-Nikodým property (briefly, RNP) with respect to $(\Omega,\mathcal{A},\mu)$, see [57] and [46] for the notion of RNP with respect to a finite measure space and a measure space, respectively.

Theorem 5.3[47, 46] Every B'-valued w^*-μ-measurable function on $(\Omega,\mathcal{A},\mu)$ must be w^*-μ-equivalent to a B'-valued μ-measurable function on $(\Omega,\mathcal{A},\mu)$ iff B' has RNP with respect to $(\Omega,\mathcal{A},\mu)$.

Theorem 5.4 Let $f: (\Omega,\mathcal{A},\mu) \times (B, \|\cdot\|) \to K$ be a random operator such that $f(\omega,\cdot): B \to K$ is a continuous linear functional for each ω in Ω, and B' has RNP with respect to $(\Omega,\mathcal{A},\mu)$. Then there exists a.e. uniquely a B'-valued random variable V on $(\Omega,\mathcal{A},\mu)$ such that $f(\omega,b) = \langle V(\omega),b \rangle$ a.e. for all b in B.

Proof Define $\hat{f}: L(\mu,B) \to L(\mu,K)$ as follows: since Proposition 2.1 tell us that every p in $L(\mu,B)$ must include a B'-valued $\mathcal{A}$-random variable p^0 on Ω that is unique a.e., then $f(\cdot,p^0(\cdot)): (\Omega,\mathcal{A},\mu) \to K$ must be K-valued $\mathcal{A}$-measurable, we define $\hat{f}(p)$ to be the μ-equivalence class determined by $f(\cdot,p^0(\cdot))$. By Remark 3.3 there exists $\xi^0 \in L_0^+(\Omega)$ such that $|f(\omega,b)| \leq \xi^0(\omega) \cdot \|p^0(\omega)\|$ a.e. for all p in $L(\mu,B)$ (p^0 is as above) by noticing p^0 is separably-valued. Then $\hat{f}$ is clearly an a.e. bounded E-random linear functional such that $|\hat{f}(p)| \leq \xi \cdot \|p\|$ for all p in $L(\mu,B)$, where ξ is the μ-equivalence class of ξ^0. Now by Theorem 5.2 there exists q in $L(\mu,B')$ such that $\hat{f}(p) = \langle q,p \rangle$ for all p in $L(\mu,B)$, then Proposition 2.1 again produces a representative V of q such that V is a B'-valued $\mathcal{A}$-random variable on Ω. By the definition of $\hat{f}$, V is just the desired!

This completes the proof.

Combining the principal results in this section with those in [43, 67] together with difficult open problems in [60], we present the following:

Open Problem 5.1 Prove or disprove: a Banach space B is such that every B-valued weak random element defined on an arbitrary probability space $(\Omega,\mathcal{A},\mu)$ must be weakly equivalent to a B-valued random variable on $(\Omega,\mathcal{A},\mu)$ iff B is a weak Asplund space.

Remark 5.3 There are the following difficult open problems in [60]: Is a closed subspace of a weak Asplund space a weak Asplund space? Is the product of two weak Asplund spaces a weak Asplund space? The fundamental importance of Open Problem 5.1 is both in random functional analysis itself and in particular in that an affirmative answer to it yields a simultaneous solution of the above two problems. It is well known that a weakly compactly generated Banach space or a conjugate space with the RNP, e.g., see [43, 67], has the weak-equivalence-property as in Open Problem 5.1, while the two classes of spaces are both weak Asplund spaces.

6 Characterizations of Random Reflexive Spaces

In this section, let $(B, \|\cdot\|)$ be a Banach space over K; $L(\mu, B)$ the same as in Section 5; for Lebesgue-Bochner function spaces $L^p(\mu, B)$ $(1 \le p \le +\infty)$, see [57] for details; q the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, the ordinary p-norm on $L^p(\mu, K)$ is denoted by $\|\cdot\|_p$.

Throughout this section, $(S, \mathscr{X})$ always denotes a given complete E-RN module over K with base $(\Omega, \mathscr{A}, \mu)$, $(S^*, \mathscr{X}^*)$ its E-random conjugate space, $(S^{**}, \mathscr{X}^{**})$ the E-random conjugate space of $(S^*, \mathscr{X}^*)$.

Denote $L^p(S) = \{g \in S \mid \|X_g\|_p < +\infty\}$, $\|\cdot\|_p : L^p(S) \to [0, +\infty)$ is defined by $\|g\|_p = \|X_g\|_p$ for all $g \in L^p(S)$, then $(L^p(S), \|\cdot\|_p)$ is a Banach space; Similarly, one can understand $(L^q(S^*), \|\cdot\|_q)$.

As was said in [57, 43], representation of the dual of Lebesgue-Bochner function spaces $L^p(\mu, B)$ $(1 \le p \le +\infty)$ has a long and distinguished history, see [46, 43] and their references, indeed, it is [46, Theorem 3.1] as a generalization of Theorem 6.1 below that captures the essence of all representation theorems of the dual of $L^p(\mu, B)$.

Theorem 6.1[48] Define $T : L^q(S^*) \to (L^p(S))'$ (namely the ordinary conjugate space or dual of $L^p(S)$) as follows: for each $f \in L^q(S^*)$, T_f (denoting $T(f)$) $: L^p(S) \to K$ by $T_f(g) = \int_\Omega f(g)\, d\mu$ for all $g \in L^p(S)$, where $1 \le p < +\infty$. Then we have $L^q(S^*)$ is isomorphically isometric to $(L^p(S))'$ under the canonical mapping T, briefly $L^q(S^*) \cong (L^p(S))'$.

Theorem 6.1 can be derived from the following two lemmas in whose proofs we can, without loss of generality, suppose $(\Omega, \mathscr{A}, \mu)$ is a probability space.

Lemma 6.1[48] T is isometric.

Proof Clearly, $|T_f(g)| = \left|\int_\Omega f(g)\, d\mu\right| \le \int_\Omega X_f^* \cdot X_g\, d\mu \le \|f\|_q \cdot \|g\|_p$ for all f in $L^q(S^*)$ and all g in $L^p(S)$, and thus $\|T_f\| \le \|f\|_q$ for all f in $L^q(S^*)$.

On the other hand, let $S(1) = \{g \in S \mid X_g \le 1\}$, then there exists a sequence $\{g_n\}$ in $S(1)$ such that $\{|f(g_n)| \mid n \in N\} \nearrow X_f^*$ since $\{|f(g)| \mid g \in S(1)\}$ is directed. For a ξ in $L(\mu, K)$, we always denote $|\xi|^{-1} \cdot \xi$ by $\mathrm{sgn}(\xi)$.

When $p = 1$ $(q = +\infty)$, for an $\epsilon > 0$, let $A(\epsilon) = [X_f^* > \|f\|_\infty - \epsilon]$, since $\mu(A(\epsilon)) > 0$, there exists some n_0 in N such that $\mu(B(\epsilon)) > 0$, where $B(\epsilon) = [|f(g_{n_0})| > \|f\|_\infty - \epsilon]$, take $g_\epsilon = \frac{1}{\mu(B(\epsilon))} \cdot \overline{\mathrm{sgn}(f(g_{n_0}))} \cdot I_{B(\epsilon)} \cdot g_{n_0}$, then $\|g_\epsilon\|_1 \le 1$ and $|T_f(g_\epsilon)| > \|f\|_\infty - \epsilon$, and hence $\|T_f\| \geq \|f\|_\infty$.

When $p > 1$, for all n in N, $\int_\Omega |f(g_n)|^q d\mu = \int_\Omega |f(g_n)|^{q-1} \cdot \overline{\text{sgn}(f(g_n))} f(g_n) d\mu = T_f(|f(g_n)|^{q-1} \overline{\text{sgn}(f(g_n))} g_n) \leq \|T_f\| \left(\int_\Omega |f(g_n)|^q d\mu\right)^{\frac{1}{p}}$. Then $\|f(g_n)\|_q \leq T_f$, again by Levy's theorem one can have $\|f\|_q = \|X_f^*\|_q \leq \|T_f\|$.

This completes the proof.

Lemma 6.2[48] T is surjective.

Proof Let F be a given element in $(L^p(S))'$ and $g \in L^\infty(S)$, define the two vector measures $G_g: \mathscr{A} \to K$ and $G: \mathscr{A} \to (L^\infty(S))'$ as follows:

$G_g(E) = F(\tilde{I}_E \cdot g)$ for all E in $\mathscr{A}$;

$G(E)(h) = F(\tilde{I}_E \cdot h)$ for all h in $L^\infty(S)$ and all E in $\mathscr{A}$.

Since $|G(E)(h)| \leq \|F\| \|\tilde{I}_E \cdot h\|_p \leq \|F\| \cdot \|h\|_\infty \cdot \|\tilde{I}_E\|_p$ for all E in $\mathscr{A}$ and all h in $L^\infty(S)$, G and G_g are both countably additive. Again since for a finite partition $\{E_i | 1 \leq i \leq n\}$ of Ω to $\mathscr{A}$, and for a finite subset $\{h_i | 1 \leq i \leq n\}$ of the closed unit ball of $L^\infty(S)$, we have $\left|\sum_{i=1}^n G(E_i)(h_i)\right| = \left|F\left(\sum_{i=1}^n \tilde{I}_{E_i} h_i\right)\right| \leq \|F\| \left\|\sum_{i=1}^n \tilde{I}_{E_i}\right\|_p = \|F\|$, likewise, we also have $\sum_{i=1}^n |G_g(E_i)| = \sum_{i=1}^n G(E_i)(\overline{\text{sgn}(G_g(E_i))} g) \leq \|F\| \|g\|_\infty$. Thus $|G| \leq \|F\|$ and $|G_g| \leq \|F\| \cdot \|g\|_\infty$, where $|G|$ and $|G_g|$ denote the total variations of G and G_g, respectively. Clearly, G and G_g are both μ-continuous.

According to the classical Radon-Nikodym theorem, for an arbitrary g in $L^\infty(S)$ there exists uniquely some $f(g)$ in $L^1(\mu, K)$ such that $G_g(E) = \int_E f(g) d\mu$ for all E in $\mathscr{A}$, and such that $|G_g|(E) = \int_E |f(g)| d\mu$ for all E in $\mathscr{A}$. Thus we can have a mapping $f: L^\infty(S) \to L^1(\mu, K)$ such that the following hold:

1) $f(\alpha \cdot g_1 + \beta \cdot g_2) = \alpha \cdot f(g_1) + \beta \cdot f(g_2)$ for all α, β in K and all g_1, g_2 in $L^\infty(S)$;

2) $f(\varphi \cdot g) = \varphi \cdot f(g)$ for all simple elements φ in $L(\mu, K)$ and all $g \in L^\infty(S)$.

Now we can claim that $f(\varphi \cdot g) = \varphi \cdot f(g)$ for all φ in $L^\infty(\mu, K)$ and all g in $L^\infty(S)$. In fact, there always exists a sequence $\{\varphi_n\}$ of simple elements in $L(\mu, K)$ such that $\|\varphi_n - \varphi\|_\infty \to 0$, and thus $\|f(\varphi \cdot g) - f(\varphi_n \cdot g)\|_1 = |G_{(\varphi - \varphi_n) \cdot g}| \leq \|F\| \cdot \|\varphi - \varphi_n\|_\infty \cdot \|g\|_\infty \to 0$, again observing $\|\varphi \cdot f(g) - \varphi_n \cdot f(g)\|_1 \leq \|\varphi - \varphi_n\|_\infty \cdot \|f(g)\|_1 \to 0$, yields $f(\varphi \cdot g) = L^1 - \lim_{n\to\infty} f(\varphi_n \cdot g) = L^1 - \lim_{n\to\infty}(\varphi_n \cdot f(g)) = \varphi \cdot f(g)$. We will also claim that $\{|f(g)| | g \in S(1)\}$ is directed as follows.

In fact, for any two elements g_1, g_2 in $S(1)$, let $E_1 = [|f(g_1)| < |f(g_2)|]$ and $E_2 = E_1^c$, then $|f(g_1)| \vee |f(g_2)| = I_{E_1} \cdot |f(g_2)| + I_{E_2} \cdot |f(g_1)| = f(\overline{\text{sgn}(f(g_2))} \cdot I_{E_1} \cdot g_2 + \overline{\text{sgn}(f(g_1))} \cdot I_{E_2} \cdot g_1) = f(g_3) = |f(g_3)|$, where $g_3 = \overline{\text{sgn}(f(g_2))} \cdot I_{E_1} \cdot g_2 + \overline{\text{sgn}(f(g_1))} \cdot I_{E_2} \cdot g_1$. Then g_3 is clearly in $S(1)$, and hence $\{|f(g)| | g \in S(1)\}$ is directed. Denote $\vee\{|f(g)| | g \in S(1)\}$ by ξ, then there exists a sequence $\{g_n | n \in N\}$ in $S(1)$ such that

$\{ \| f(g_n) \| \, n \in N \} \nearrow \xi$. By Levy's theorem $\| \xi \|_1 = \lim_{n\to\infty} \| f(g_n) \|_1 = \lim_{n\to\infty} \| G_{g_n} \| \le \| G \| \le \| F \| < +\infty$, thus $\xi \in L^1(\mu, K)$.

For any g in $L^\infty(S)$ and an $\epsilon > 0$, since $(X_g + \epsilon)^{-1} \le \frac{1}{\epsilon}$, $(X_g + \epsilon)^{-1}$ is in $L^\infty(\mu, K)$, and hence $| f((X_g + \epsilon)^{-1} \cdot g) | = (X_g + \epsilon)^{-1} \cdot | f(g) | \le \xi$, namely $| f(g) | \le \xi \cdot (X_g + \epsilon)$ for $\epsilon > 0$, one can have $| f(g) | \le \xi \cdot X_g$ for all g in $L^\infty(S)$.

It is obvious that f is continuous relative to the (ϵ, λ)-linear topology on $L^\infty(S)$, and that $L^\infty(S)$ is dense in S with respect to the (ϵ, λ)-linear topology on S, and thus f can be uniquely extended to S, this extension is still denoted by f. It is also easy to see that $| f(g) | \le \xi \cdot X_g$ for all g in S, i.e., $f \in S^*$. In fact, $f \in L^1(S^*)$ since $X_f^* \le \xi$.

According to the definitions of G_g and f, one can have $F(g) = G_g(\Omega) = \int_\Omega f(g)\, d\mu$ for all g in $L^\infty(S)$. Put $E_n = [X_f^* \le n]$ and $f_n = I_{E_n} \cdot f$ for each $n \in N$, then $f_n \in L^q(S^*)$, $X_{f_n}^* = I_{E_n} \cdot X_f^*$ and $\int_{E_n} f(g)\, d\mu = F(I_{E_n} \cdot g)$ for all $g \in L^\infty(S)$ and all $n \in N$, namely $T_{f_n}(g) = F(I_{E_n} \cdot g)$ for each n in N and all g in $L^\infty(S)$. Since $L^\infty(S)$ is dense in $L^p(S)$ in the norm $\| \cdot \|_p$, then $T_{f_n}(g) = F(I_{E_n} \cdot g)$ for all g in $L^p(S)$, and hence $\| T_{f_n} \| \le \| F \|$ for all n in N. Since $\left(\int_\Omega (X_f^*)^q\, d\mu \right)^{\frac{1}{q}} = \lim_{n\to\infty} \| f_n \|_q = \lim_{n\to\infty} \| T_{f_n} \| \le \| F \| < +\infty$, then $f \in L^q(S^*)$ and $F(g) = \lim_{n\to\infty} F(I_{E_n} \cdot g) = \lim_{n\to\infty} \int_{E_n} f(g)\, d\mu = \int_\Omega f(g)\, d\mu$ for all g in $L^p(S)$, i.e., $F = T_f$.

This completes the proof.

Remark 6.1 Theorem 6.1 holds also for arbitrary noncomplete E-RN modules, and it can be extended to the case when $(\Omega, \mathscr{A}, \mu)$ is an arbitrary measure space by the linear projection limit method, see [46, Theorem 3.1] for details. [46, Theorem 3.1] unifies all representation theorems of the dual of $L^p(\mu, B)$ currently available, e.g., which is easy to see by taking $S = L(\mu, B)$.

Definition 6.1[47] The canonical embedding $J: S \to S^{**}$ is defined by $J(p)(f) = f(p)$ for all p in S and all f in S^*. $(S, \mathscr{X})$ is called random reflexive if J is surjective.

Theorem 6.2[48] Let p be a given positive number such that $1 < p < +\infty$. Then $(S, \mathscr{X})$ is random reflexive iff $(L^p(S), \| \cdot \|_p)$ is a reflexive Banach space.

Proof Denote the classical canonical embedding from $L^p(S)$ into $(L^p(S))''$ by j, then Necessity follows immediately from Theorem 6.1.

(Sufficency) We only need to prove J surjective. Let F be in S^{**}, for each $n \in N$, set $E_n = [n - 1 \le X_F^{**} < n]$, then $\sum_{n=1}^\infty \mu(E_n) = \mu(\Omega) = 1$ (without loss of generality, we can suppose $\mu(\Omega) = 1$). From $I_{E_n} \cdot F \in L^r(S^{**}) \equiv j(L^r(S))$ one can see that there exists $g_n \in L^r(S)$ such that $j(g_n) = I_{E_n} \cdot F$, i.e., $\int_\Omega f(g_n)\, d\mu = j(g_n)(f) = \int_\Omega (I_{E_n} \cdot F)(f)\, d\mu$ for all $f \in L^q(S^*)$. For any E in $\mathscr{A}$ and f in $L^q(S^*)$, since $I_E \cdot f$ still belongs to $L^q(S^*)$, we can also, of course, have $\int_E f(g_n)\, d\mu = \int_\Omega (I_E f)(g_n)\, d\mu = \int_\Omega (I_{E_n} F)(I_E f)\, d\mu = \int_E (I_{E_n} F)(f)\, d\mu$. Thus $f(g_n) = (I_{E_n} \cdot F)(f)$ for all f in $L^q(S^*)$, and hence also $f(g_n) = (I_{E_n} \cdot F)(f)$ for all f in S^* since $L^q(S^*)$ is dense in S^* in the (ϵ, λ)-linear topology on S^*, namely $J(g_n) = I_{E_n} \cdot F$ for all n in N.

Put $\hat{g}_k = \sum_{n=1}^{k} I_{E_n} g_n$ for each k in N, then it is easy to see $\{\hat{g}_k \mid k \in N\}$ is a Cauchy sequence in $(S, \mathscr{X})$, hence convergent to some g in S. From the continuity of J we can have $J(g) = \lim_{k\to\infty} J(\hat{g}_k) = \lim_{k\to\infty} \sum_{n=1}^{k} (I_{E_n} F) = \left(\sum_{n=1}^{\infty} I_{E_n}\right) F = F$.

This completes the proof.

In [47], we have obtained the following:

Theorem 6.3[47] $L(\mu, B)$ is random reflexive iff B is a reflexive Banach space.

These characterizations—Theorems 6.2 and 6.3 concerning random reflexivity are both described in terms of reflexivity of related Banach spaces. Following are several important intrinsic characterizations of random reflexive spaces.

First, let us recall from [49]: a subset G in S is called strongly convex if $\xi \cdot g_1 + \eta \cdot g_2$ still belongs to G for all g_1, g_2 in G and all ξ, η in $L^+(\mu)$ such that $\xi + \eta = 1$. Concerning the relation between random distances and ordinary L^p-distances, we have the following:

Theorem 6.4 Let $1 \le p \le +\infty$, G a strongly convex closed set of $(S, \mathscr{X})$ such that $\theta \in G$, and $g \in L^p(S)$. Denote $\wedge\{X_{g-g_1} \mid g_1 \in G\}$ by $X_{g,G}$, and $\inf\{\|g - g_1\|_p \mid g_1 \in L^p(G)\}$ by dist $(g, L^p(G))$, where $L^p(G) = \{g_1 \in G \mid \|X_{g_1}\|_p \equiv \|g_1\|_p < +\infty\}$. Then we have $\|X_{g,G}\|_p = \text{dist}(g, L^p(G))$.

The following theorem gives the relation between pointwise proximinality (namely RM-proximinality) and ordinary proximinality.

Theorem 6.5 Let G be a closed submodule of $(S, \mathscr{X})$ and $1 \le p \le +\infty$. Then we have G is pointwise (or, RM-) proximinal in $(S, \mathscr{X})$ iff $L^p(G)$ is $\|\cdot\|_p$-proximinal in $(L^p(S), \|\cdot\|_p)$, this relation still holds for an arbitrary strongly convex closed subset whenever $1 \le p < +\infty$.

Corollary 6.1[50] Let Y be a closed subspace of B and $1 \le p \le +\infty$. Then $L(\mu, Y)$ is pointwise proximinal in $L(\mu, B)$ iff $L^p(\mu, Y)$ is $\|\cdot\|_p$-proximinal in $L^p(\mu, B)$.

Combining measurable selection theorems[55] and Corollary 6.1 yields the following:

Theorem 6.6[51] Let Y be a separable subspace of B and $1 \le p \le +\infty$. Then Y is proximinal in B iff $L^p(\mu, Y)$ is $\|\cdot\|_p$-proximinal in $L^p(\mu, B)$.

Returning to characterizations of random reflexive spaces, we have:

Theorem 6.7 The following statements are equivalent to each other:

1) $(S, \mathscr{X})$ is random reflexive;

2) Every strongly convex closed subset of S is pointwise proximinal in $(S, \mathscr{X})$;

3) Every closed submodule of S is pointwise proximinal in $(S, \mathscr{X})$;

4) The random norm of every a.e. bounded E-random linear functional on S is attainable on $S(1)=\{g\in S \mid X_g\leq 1\}$.

Theorem 6.7 shows that the famous classical James theorem—namely a Banach space is reflexive iff every bounded linear functional on the space can attain its norm on the closed unit ball of the space, can be extended to a complete E-RN module. It is fairly well known that a Banach space is reflexive iff its closed unit ball is weakly compact, however, the deep theorem rarely holds for general complete E-RN modules.

The following theorem is a random version of the classical James theorem mentioned above.

Theorem 6.8 Let $f:(\Omega,\mathcal{A},\mu)\times(B,\|\cdot\|)\to K$ be a random operator such that $f(\omega,\cdot):B\to K$ is a continuous linear functional for each ω in Ω, B an reflexive Banach space, and ξ be the essential supremum of the set $\{f(\cdot,b) \mid b\in B \text{ and } \|b\|\leq 1\}$. Then there exists a B-valued random variable p^0 on $(\Omega,\mathcal{A},\mu)$ such that $\|p^0(\omega)\|=1$ and $\xi(\omega)=f(\omega,p^0(\omega))$ for almost all ω in Ω.

Proof Define $\hat{f}:L(\mu,B)\to L(\mu,K)$ as in Theorem 5.4, then $\hat{f}\in(L(\mu,B))^*$, and $\hat{\xi}=$ the μ-equivalence class of ξ is just $X_{\hat{f}}^*$. Then it follows immediately from Theorems 6.3 and 6.7 that there exists $p\in L(\mu,B)$ such that $\|p\|\leq 1$ and $\hat{f}(p)=\hat{\xi}$. It is easy to check that $\|p\|=1$. Chosing an arbitrary representative p^0 of p such that p^0 is random variable. Then p^0 is just the desired.

This completes the proof.

Theorem 6.9 below is a random approximation result.

Theorem 6.9 Let $(B,\|\cdot\|)$ be a reflexive Banach space over K, g^0 a B-vallued random variable on $(\Omega,\mathcal{A},\mu)$, and M a closed convex subset of B. Then there exists an M-valued random variable h^0 on $(\Omega,\mathcal{A},\mu)$ such that $\|g^0(\omega)-h^0(\omega)\|=\mathrm{dist}(g^0(\omega),M)$ for almost all ω in Ω.

Proof Let ξ be the μ-equivalence class of $\mathrm{dist}(g^0(\cdot),M)$ and $L(\mu,M)$ be the set of all μ-equivalence classes of M-valued μ-measurable functions on $(\Omega,\mathcal{A},\mu)$, then $L(\mu,M)$ is a strongly convex closed subset of the random reflexive E-RN module $L(\mu,B)$, and thus from Theorem 6.7 there exists h in $L(\mu,M)$ such that $\|g-h\|=X_{g,L(\mu,M)}=\wedge\{\|g-h_1\| \mid h_1\in L(\mu,M)\}$, where g is the μ-equivalence class of g^0. It is easy to check that $\xi=X_{g,L(\mu,M)}$, and thus an arbitrary representative h^0 of h such that h^0 is random variable, is just the desired!

This completes the proof.

7 E-random Seminormed Modules and E-random Dualities.

Just as functional analysis needs these frameworks such as linear topological spaces and in particular locally convex spaces, so random metric theory does similar counterparts of the above mentioned notions. Even at the outset we already realized this point. After the author gave Theorems 3.1 and 3.2 in [31], motivated by the method of the author's constructing F-random metric and F-random norm Gong made use of the technique used in Theorem 3.2 to consider the corresponding problem of random elements valued in a locally convex spaces so that he first studied the so-called F-random lovally convex spaces in [35]; slightly later You Zhao-yong, Gong Fu-zhou and the author further considered random elements valued in linear topological spaces and the so-called F-random linear topological spaces in [38]; and in [41] we also mentioned similar things in the course of studying random weak topologies. In [32], the author gave a number of substantial results that amount to the main content of [34]. In subsequent years the subject lay dormant because of lacking means. Recently motivated by the study of E-RN modules the subject has obtained the further developments in [47]. The purpose of this section is to briefly introduce several recent results on the subject and to standardize the related terminologies. Since the linear topology of a linear topological space (resp., a locally convex space) is always generated by a family of pseudonorms (resp., seminorms)[67, 68], sometimes a linear topological space (resp., a locally convex space) is also called a pseudonormed linear space (resp., a seminormed linear space)[67], this formulation can easily extend to a random version so that we can easily have the following notion.

Definition 7.1 An ordered pair $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ is called an E-random pseudonormed linear space (briefly, an E-$RPNL$ space) over K with base $(\Omega, \mathscr{A}, \mu)$ if S is a linear space over K, Γ is an indexing set, and for each d in Γ $\mathscr{X}^d$ is a mapping from S to $L^+(\mu)$ such that, denoting $\mathscr{X}^d(p)$ by X_p^d, for all d in Γ, all p, q in S and all α in K, the following hold:

1) $X_p^d = 0$ for each d in Γ iff $p = \theta$;
2) $X_{\alpha p}^d = \alpha X_p^d$;
3) for each d in Γ there exists some e in Γ such that $X_{p+q}^d \leq X_p^e + X_q^e$.

In addition, if there exists another mapping $*$: $L(\mu, K) \times S \rightarrow S$ such that the following two items are also satisfied:

4) $(S, *)$ is a left module over the algebra $L(\mu, K)$;
5) $X_{\xi * p}^d = \xi\ X_p^d$ for all d in Γ, all ξ in $L(\mu, K)$ and all p in S.

Then the triple $(S, \{\mathscr{X}^d\}_{d\in\Gamma}, *)$ is called an E-random pseudonormed module (briefly, an E-RPN module) over K with base $(\Omega, \mathscr{A}, \mu)$. Furthermore, the E-$RPNL$ space (resp., E-RPN module) is called an E-random seminormed linear space (resp., an E-random seminormed module), briefly called an E-$RSNL$ space(resp., an E-RSN module) if for each d in Γ, $\mathscr{X}^d$ is always an E-random seminorm, namely the above 3) can be strengthened as: 3)' $X_{p+q}^d \leq X_p^d + X_q^d$ for all d in Γ and all p, q in S. For the same reason as in [49], we can abbrieviate $(S, \{\mathscr{X}^d\}_{d\in\Gamma}, *)$ to $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$, and $\xi * p$ to $\xi \cdot p$ or ξp whenever $*$ is understood.

Remark 7.1 It is easy to see that when $\{\mathscr{X}^d\}_{d\in\Gamma}$ degenerates to a singleton $\{\mathscr{X}\}$, then the E-$RPNL$ space (resp., the E-RPN module) $(S, \mathscr{X})$ is just an E-RN space (resp., an E-RN module). We prefer the terminologies "an E-$RPNL$ space" and "an E-

$RSNL$ space "to those "an E-random linear topological space "and "an E-random locally convex space "because the (ϵ,λ)-topologies for these spaces are rarely locally convex topologies and because what play key roles is the E-random-pseudonorm structures and E-random seminorm structures instead of the (ϵ,λ)-topological structures.

Proposition 7.1 Let $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-$RPNL$ space over K with base $(\Omega,\mathscr{A},\mu)$, and $\mathscr{F}(\mathscr{A})$ be the set-family $\{A\in\mathscr{A}\mid 0<\mu(A)<+\infty\}$. For an A in $\mathscr{F}(\mathscr{A})$, d in Γ, $\epsilon>0$, and $0<\lambda<\mu(A)$, denote $N_\theta(A,d,\epsilon,\lambda)=\{p\in S\mid \mu\{\omega\in A\mid X_p^d(\omega)<\epsilon\}>\mu(A)-\lambda\}$; let $\mathscr{U}_\theta(A,d)=\{N_\theta(A,d,\epsilon,\lambda)\mid \epsilon>0, 0<\lambda<\mu(A)\}$ for each $A\in\mathscr{F}(\mathscr{A})$ and each d in Γ. Then $\mathscr{U}_\theta:=\cup\{\mathscr{U}_\theta(A,d)\mid A\in\mathscr{F}(\mathscr{A}), d\in\Gamma\}$ forms a subbase at the null θ of S for some Hausdorff linear topology on S (called the (ϵ,λ)-linear topology), in particular $L(\mu,K)$, as a special E-RN module, is a topological algebra in its (ϵ,λ)-linear topology, in addition if $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ is an E-RPN module, then it is a topological module over the topological algebra $L(\mu,K)$ in the (ϵ,λ)-linear topology of S.

In our work, the topology for an E-$RPNL$ space always means its (ϵ,λ)-linear topology unless otherwise stated.

Let $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-$RPNL$ space over K with base $(\Omega,\mathscr{A},\mu)$ and $\mathscr{F}(\Gamma)$ be the family of all finite subsets of Γ. Then $(S,\{\mathscr{X}^F\}_{F\in\mathscr{F}(\Gamma)})$ is still an E-$RPNL$ space over K with base $(\Omega,\mathscr{A},\mu)$, where $\mathscr{X}^F: S\to L^+(\mu)$ is defined by $X_p^F=\sum_{d\in F}X_p^d$ for all p in S and all F in $\mathscr{F}(\Gamma)$. Let $\{A_n\mid n\in N\}$ be an arbitrarily chosen sequence of μ-measurable subsets of Ω such that $\{A_n\}$ is a partition of Ω, $\{\xi_n\}$ an arbitrarily chosen sequence in $L^+(\mu)$, and $\{F_n\}$ an arbitrarily chosen sequence in $\mathscr{F}(\Gamma)$, define $\mathscr{X}: S\to L^+(\mu)$ by $X_p=\sum_{n=1}^{\infty}\tilde{I}_{A_n}\cdot\xi_n\cdot\mathscr{X}_p^{F_n}$ for all p in S, then $\mathscr{X}$ is a continuous mapping from S to $L(\mu,R)$, denote the set of all $\mathscr{X}$ obtained in such a manner by by $C(\Gamma)$. Then $(S,C(\Gamma))$ is still an E-$RPNL$ space over K with base $(\Omega,\mathscr{A},\mu)$; and an E-RPN module if $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ is an E-RPN module. In particular $(S,C(\Gamma))$ is also so if $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ is an E-$RSNL$ space (resp., an E-RSN module).

Definition 7.2 Let $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-$RPNL$ space over K with base $(\Omega,\mathscr{A},\mu)$. An E-random linear functional $f: S\to L(\mu,K)$ is called a. e. bounded if there exists some $\mathscr{X}$ in $C(\Gamma)$ such that $|f(p)|\leq X_p$ for all p in S. Denote by S^* the left module over $L(\mu,K)$ of all a. e. bounded E-random linear functionals on S with the module multiplication operation as in Definition 4.3. Clearly, when S is an E-RN space, Definition 7.2 returns to Definition 4.3. Definition 7.2 is distinct from that in [34]. For the advantages of Definition 7.2, see Theorem 7.2.

Theorem 7.1([34]) Let $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-$RSNL$ space over K with base $(\Omega,\mathscr{A},\mu)$, and p be a nonzero element in S. Then there exists f in S^* such that $f(p)\neq 0$.

Theorem 7.2 Let $(S,\{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-RSN module over K with base $(\Omega,\mathscr{A},\mu)$. Then an E-random linear functional f on S is a. e. bounded iff f is a continuous module homomorphism from S to $L(\mu, K)$.

Remark 7.2 Theorem 7.2 is due to the author, whose proof together with whose generalization is quite long and will be given in the forthcoming paper.

Definition 7.3 Let $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ be the same as in Definition 7.2. A subset M of S is called bounded if M is bounded in the (ϵ, λ)-linear topology, and further called a.e. bounded if $\vee\{X_p^d \mid p \in M\}$ is still in $L^+(\mu)$ for each d in Γ. Clearly M is a.s. bounded iff $\vee\{X_p \mid p \in M\}$ is in $L^+(\mu)$ for each $\mathscr{X}$ in $C(\Gamma)$.

Theorem 7.3 Let $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ be the same as in Theorem 7.2. Then a subset M of S is bounded (resp., a.e. bounded) iff for each f in S^*, $\{f(p) \mid p \in M\}$ is bounded (resp., a.e. bounded) in $L(\mu, K)$.

Proof As was pointed out in [49], we can suppose $\mu(\Omega) = 1$. It is also easy to see that the (ϵ, λ)-linear topology determined by $\{\mathscr{X}^d\}_{d\in\Gamma}$ is identical with that determined by $C(\Gamma)$, and thus the proof follows at once from [34, Theorem 3.3].

This completes the proof.

Definition 7.4[52] Let S_1 and S_2 be the two left module over the algebra $L(\mu, K)$ and $\langle\cdot, \cdot\rangle: S_1 \times S_2 \to L(\mu, K)$ be such that the following hold:

1) $\langle s_1, \cdot\rangle: S_2 \to L(\mu, K)$ is a module homomorphism for each s_1 in S_1;
2) $\langle\cdot, s_2\rangle: S_1 \to L(\mu, K)$ is a module homomorphism for each s_2 in S_2;
3) $\langle s_1, s_2\rangle = 0$ for all s_2 in S_2 implies $s_1 = \theta$;
4) $\langle s_1, s_2\rangle = 0$ for all s_1 in S_1 implies $s_2 = \theta$.

Then S_1 and S_2 is said to be in duality with respect to $\langle\cdot, \cdot\rangle$, or, $\langle S_1, S_2\rangle$ is called an E-random duality over K with base $(\Omega, \mathscr{A}, \mu)$. Denote $\{|\langle\cdot, s_2\rangle| \mid s_2 \text{ in } S_2\}$ by $\sigma(S_1, S_2)$, and $\{|\langle s_1, \cdot\rangle| \mid s_1 \text{ in } S_1\}$ by $\sigma(S_2, S_1)$, then $(S_1, \sigma(S_1, S_2))$ and $(S_2, \sigma(S_2, S_1))$ are both E-RSN modules, called $\sigma(S_1, S_2)$-RSN module and $\sigma(S_2, S_1)$-RSN module, respectively. The (ϵ, λ)-linear topologies of $(S_1, \sigma(S_1, S_2))$ and $(S_2, \sigma(S_2, S_1))$ are called the weak (ϵ, λ)-linear topopogies of S_1 and S_2, respectively.

The following theorem is key to establish the theory of E-random dualities.

Theorem 7.4[52] Let S be a left module over the algebra $L(\mu, K)$, and $f_1, f_2, \cdots, f_n$ and g be module homomorphisms from S to $L(\mu, K)$ such that $\bigcap_{k=1}^{n}\mathrm{Ker}(f_k) \subset \mathrm{Ker}(g)$, where $\mathrm{Ker}(f_k) = \{p \in S \mid f_k(p) = 0\}$ for all k in $\{1, 2, \cdots, n\}$ and $\mathrm{Ker}(g) = \{p \in S \mid g(p) = 0\}$. Then there exist $\xi_1, \xi_2, \cdots, \xi_n$ in $L(\mu, K)$ such that $g(p) = \sum_{i=1}^{n} \xi_i \cdot f_i(p)$ for all p in S.

Proof We proceed as follows by the induction method, and can, without loss of generality, suppose $\mu(\Omega) = 1$.

1) when $n = 1$: since S is an $L(\mu, K)$-module, it is easy to see that $\{|f_1(p)| \mid p \in S\}$ is directed, and hence there exists a sequence $\{p_n\}$ in S such that $\{|f_1(p_n)| \mid n \in N\} \nearrow \xi := \vee\{|f_1(p)| \mid p \in S\}$ (ξ is in general in $\bar{L}(\mu, R)$ and $\xi \geq 0$). Set $B = [\xi > 0]$ and $B_i = [|f_1(p_i)| > 0]$ for each i in N, then $B = \bigcup_{i=1}^{\infty} B_i$ and $B_n \subset B_{n+1}$ for each n in N, again set $B_0 = \emptyset$, $A_n =$

$B_n \backslash B_{n-1}$ for each $n \geq 1$, then $A_i \cap A_j = \not\subset$ $(i = j)$ and $B = \sum_{n=1}^{\infty} A_n$.

Putting $q_i = I_{A_i} \cdot (f_1(p_i))^{-1} \cdot p_i$ yields $f_1(q_i) = I_{A_i} \cdot (f_1(p_i))^{-1} \cdot f_1(p_i) = I_{A_i} \cdot I_{B_i} = I_{A_i}$ for each i in N, hence $f_1(I_{A_i} \cdot p - f_1(p) \cdot q_i) = I_{A_i} \cdot f_1(p) - f_1(p) \cdot I_{A_i} = 0$, this shows $I_{A_i} \cdot p - f_1(p) \cdot q_i \in \mathrm{Ker}(f_1) \subset \mathrm{Ker}(g)$ for each p in S, namely $g(I_{A_i} \cdot p - f_1(p) \cdot q_i) = 0$, therefore $I_{A_i} \cdot g(p) = f_1(p) \cdot g(q_i) = g(q_i) \cdot f_1(p)$ for each i in N and each p in S.

$$(7.4.1)$$

By the definition of B, one can easily see $f_1(I_{B^c} \cdot p) = I_{B^c} \cdot f_1(p) = 0$ for each p in S, and thus $I_{B^c} \cdot p \in \mathrm{Ker}(f_1) \subset \mathrm{Ker}(g)$, hence $I_{B^c} \cdot g(p) = g(I_{B^c} \cdot p) = 0$ for each p in S. Then $f_1(p) = (I_B + I_{B^c}) \cdot f_1(p) = I_B \cdot f_1(p) + I_{B^c} \cdot f_1(p) = I_B \cdot f_1(p)$ for each p in S, also yields $g(p) = I_B \; g(p) + I_{B^c} \; g(p) = I_B \; g(p) = \left(\sum_{i=1}^{\infty} I_{A_i} \right) \; g(p) = \sum_{i=1}^{\infty} I_{A_i} \; g(p) = \sum_{i=1}^{\infty} I_{A_i} \; I_{A_i} \; g(p) = \sum_{i=1}^{\infty} I_{A_i} \; (I_{A_i} \; g(p)) = \sum_{i=1}^{\infty} I_{A_i} \; g(q_i) \; f(p)$ (by (7.4.1)) for each p in S. (7.4.2).

Set $\xi_1^k = \sum_{i=1}^{k} I_{A_i} \; g(q_i)$ for each k in N, then $\{\xi_1^k \; k \in N\}$ is obviously a Cauchy sequence in $L(\mu, K)$ in the topology of convergence in measure μ from $\sum_{i=1}^{\infty} \mu(A_i) = \mu(B) \leq \mu(\Omega) = 1$, and hence convergent to some ξ_1 in $L(\mu, K)$. Thus $\xi_1 = \sum_{i=1}^{\infty} I_{A_i} \; g(q_i)$, and $g(p) = \xi_1 \cdot f_1(p)$ for each p in S by (7.4.2).

2) Suppose Theorem 7.4 holds for $n = k$. By noticing $\mathrm{Ker}(f_{k+1})$ is also an $L(\mu, K)$-module, and applying the induction Assumption 2) to $\mathrm{Ker}(f_{k+1})$ and the restrictions of f_1, $f_2, \cdots, f_k, f_{k+1}$ and g to $\mathrm{Ker}(f_{k+1})$ will yield the following relation: $g(p) = \sum_{i=1}^{k} \xi_i \; f_i(p)$ for each p in $\mathrm{Ker}(f_{k+1})$ and some $\xi_1, \xi_2, \cdots, \xi_k$ in $L(\mu, K)$ since $\bigcap_{i=1}^{k+1} \mathrm{Ker}(f_i) = (\bigcap_{i=1}^{k} \mathrm{Ker}(f_i)) \cap \mathrm{Ker}(f_{k+1}) \subset \mathrm{Ker}(g)$.

Clearly, one can easily see $\mathrm{Ker}(f_{k+1}) \subset \mathrm{Ker}\left(g - \sum_{i=1}^{k} \xi_i \; f_i\right)$, and thus there exists some ξ_{k+1} in $L(\mu, K)$ from the induction Step(1) such that $\left(g - \sum_{i=1}^{k} \xi_i \; f_i\right)(p) = \xi_{k+1} \; f_{k+1}(p)$ for each p in S, namely $g(p) = \sum_{i=1}^{k+1} \xi_i \; f_i(p)$ for each p in S.

This completes the proof.

Theorem 7.5 Let S be a linear space over K, $f: S \to L(\mu, K)$ an E-random linear functional, $\{A_i \; i \in N\}$ a countable partition of Ω to μ-measurable sets (namely, each A_i is also μ-measurable), and $X^i: S \to L^+(\mu)$ an E-random seminorm for each i in N such that $f(p) \leq \sum_{i=1}^{\infty} \tilde{I}_{A_i} \; X_p^i$ for each p in S. Then for each i in N there exists an E-random linear functional $f_i: S\to L(\mu,K)$ such that $f_i(p) \le X_p^i$ for all p in S, and such that $f(p) = \sum_{i=1}^{\infty} \tilde{I}_{A_i} f_i(p)$ for all p in S. Note: the (ϵ,λ)-linear topology for $L(\mu,K)$ is just the one of convergence locally in measure μ, namely $\{\xi_n-\xi \ n\in N\}$ converges in measure μ to 0 on every A in $\mathscr{F}(\mathscr{A})$ iff $\{\xi_n\}$ converges in the (ϵ,λ)-linear topology to ξ in $L(\mu,K)$, the series in question are clearly convergent in the (ϵ,λ)-linear topology.

Proof Let S^N be the the countable self-product of S, and $\triangle = \{(p,p,\cdots,p,\cdots)\in S^N \ p\in S\}$, then S^N and $\triangle$ are both linear spaces over K. Define $g: \triangle\to L(\mu,K)$ by $g((p,p,\cdots,p,\cdots)) = f(p)$ for all p in S, and $\mathscr{X}: S^N\to L^+(\mu)$ by $X_q = \sum_{i=1}^{\infty}\tilde{I}_{A_i} X_{p_i}^i$ for all $q = (p_1,p_2,\cdots,p_n\cdots)\in S^N$, then g is an E-random linear functional on $\triangle$, and $\mathscr{X}$ is an E-random seminorm on S^N such that $g(q) \le X_q$ for all q in $\triangle$. By Theorem 4.5 there exists an E-random linear functional $F: S^N\to L(\mu,K)$ such that $F_{\triangle} = g$ and $F(q) \le X_q$ for all q in S^N.

Now, for each $i\in N$, define an E-random linear functional $f_i: S\to L(\mu,K)$ by $f_i(p) = F(q^i)$ for all p in S, where q^i is such that $q_i^i = p$ and $q_j^i = 0$ if $i\ne j$. Clearly, $f_i(p) \le \tilde{I}_{A_i}\cdot X_p^i$ for all p in S and all i in N, therefore $f_i(p) = \tilde{I}_{A_i}\cdot f_i(p)$ and $f_i(p) \le \tilde{I}_{A_i}\cdot X_p^i \le X_p^i$ for all i in N and all p in S. Finally, it is also clear that $f(p) = \sum_{i=1}^{\infty}\tilde{I}_{A_i} f_i(p)$ for all p in S.

This completes the proof.

Similarly, one can have the following:

Theorem 7.6 Let S be a linear space over K, $f: S\to L(\mu,K)$ an E-random linear functional, and $\mathscr{X}^i: S\to L^+(\mu)$ an E-random seminorm for each i, $1\le i\le n$, such that $f(p) \le \sum_{i=1}^{n} X_p^i$ for all p in S. Then there exists an E-random linear functional $f_i: S\to L(\mu,K)$ for each i, $1\le i\le n$, such that $f(p) = \sum_{i=1}^{n} f_i(p)$ for all p in S, and such that $f_i(p) \le X_p^i$ for all p in S and each i, $1\le i\le n$.

The following is the representation theorem for weak continuous module homomorphisms:

Theorem 7.7 Let $\langle S_1, S_2\rangle$ be an E-random duality over K with base $(\Omega,\mathscr{A},\mu)$. Then for a continuous module homomorphism f from $(S_1,\sigma(S_1,S_2))$ to $L(\mu,K)$ there exist a countable partition $\{A_i \ i\in N\}$ of Ω to μ-mesurable sets and a countable set $\{q_i \ i\in N\}$ in S_2 such that $f(p) = \sum_{i=1}^{\infty}\tilde{I}_{A_i}\langle p, q_i\rangle$ for all p in S_1.

Proof According to Theorem 7.2, $f\in(S_1,\sigma(S_1,S_2))^*$, and thus from the module property of S_2 there exist a countable partition $\{A_n \ n\in N\}$ and a countable family $\{F_n\}$ of finite subsets of S_2 such that $f(p) \le \sum_{n=1}^{\infty}\tilde{I}_{A_n}\left(\sum_{q\in F_n}\langle p, q\rangle\right)$ for all p in S_1, hence by

Theorems 7.5 and 7.4 and the module property of S_2, one can complete the rest of the proof.

This completes the proof.

Remark 7.3 Let $\langle S_1, S_2\rangle$ be the same as in Theorem 7.7. Then, dually, one can also have: for a continuous module homomorphism g from $(S_2, \sigma(S_2, S_1))$ to $L(\mu, K)$ there exist similar $\{A_i \mid i\in N\}$ and a countable set $\{p_i \mid i\in N\}$ in S_1 such that $g(q) = \sum_{i=1}^{\infty}\tilde{I}_{A_i} \cdot \langle p_i, q\rangle$ for all q in S_2.

Remark 7.4 Let $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-RSN module over K with base $(\Omega, \mathscr{A}, \mu)$. Then S and its E-random conjugate space S^* form an E-random duality $\langle S, S^*\rangle$ with $\langle\cdot, \cdot\rangle: S\times S^* \to L(\mu, K)$ defined by $\langle p, f\rangle = f(p)$ for all p in S and all f in S^*, called the natural E-random duality of S and S^*. Since for a countable partition $\{A_i \mid i\in N\}$ of Ω to μ-measurable sets and a countable set $\{f_i \mid i\in N\}$ in S^*, define $f: S\to L(\mu, K)$ by $f(p) = \sum_{i=1}^{\infty}\tilde{I}_{A_i} \cdot \langle p, f_i\rangle$ for all p in S, then f is still in S^*, denoted by $\sum_{i=1}^{\infty}\tilde{I}_{A_i} \cdot f_n$, hence $(S, \sigma(S, S^*))^* = S^*$. Similarly, if S is sequently complete, in particular complete relative to the (ϵ, λ)-linear topology or weak (ϵ, λ)-linear topology of S, then we can also have $(S^*, \sigma(S^*, S))^* = S$. When $S = L(\mu, B)$, where $(B, \|\cdot\|)$ is an arbitrary normed space over K, for a countable partition $\{A_i \mid i\in N\}$ as above and a countable set $\{p_i \mid i\in N\}$ in S, clearly, there exists p in $L(\mu, B)$ such that $p = \sum_{i=1}^{\infty}\tilde{I}_{A_i} \cdot p_i$, therefore $(S^*, \sigma(S^*, S))^* = S$ still holds at this time. These observations at once lead us to the following:

Definition 7.5 Let $\langle S_1, S_2\rangle$ be an E-random duality over K with base $(\Omega, \mathscr{A}, \mu)$. S_2 is said to be standard with respect to $\langle S_1, S_2\rangle$ if for an arbitrary countable partition $\{A_i \mid i\in N\}$ of Ω to μ-measurable sets and an arbitrary countable set $\{q_i \mid i\in N\}$ in S_2 there always exist uniquely an element q in S_2 such that $\langle p, q\rangle = \sum_{i=1}^{\infty}\tilde{I}_{A_i} \cdot \langle p, q_i\rangle$ for all p in S_1, at this time $(S_1, \sigma(S_1, S_2))^* = S_2$. Similarly, one can easily define S_1 to be standard with respect to $\langle S_1, S_2\rangle$, at this time $(S_2, \sigma(S_2, S_1))^* = S_1$. Finally, if both S_1 and S_2 are standard, then $\langle S_1, S_2\rangle$ is called standard.

Theorem 7.8 Let $\langle S_1, S_2\rangle$ be a standard E-random duality. Then $(S_1, \sigma(S_1, S_2))^* = S_2$ and $(S_2, \sigma(S_2, S_1))^* = S_1$.

Remark 7.5 Let $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ be an E-RSN space over K with base $(\Omega, \mathscr{A}, \mu)$. In [32, 34], the author essentially defined an E-random linear functional $f: S\to L(\mu, K)$ to be a.e. bounded if there exist ξ in $L^+(\mu)$ and a finite subset F of Γ such that $|f(p)| \leq \xi \cdot \left(\sum_{d\in F} X_f^d\right)$ for all p in S, this leads naturally to another definition of an E-random

conjugate space. By such a definition, the author first gave Theorem 7.8 in [32] where

$\langle S_1, S_2\rangle$ can be an arbitrary E-random duality. But the definition of E-random conjugate spaces in [32, 34] can not make the sufficency of Theorem 7.2 valid. In this paper, Definition 7.2 is just motivated by Theorem 7.2. On the other hand, the requirement on standardity of an E-random duality is not too restrictive a condition as shown in Remark 7.4, in particular when the base space $(\Omega, \mathcal{A}, \mu)$ is a trivial probability space, namely $\mathcal{A} = \{\emptyset, \Omega\}$ and $\mu(\Omega) = 1$, then an E-random duality with base $(\Omega, \mathcal{A}, \mu)$ is just an ordinary duality[68], of course, it is standard!

Definition 7.6 Let $\langle S_1, S_2\rangle$ be an E-random duality over K with base $(\Omega, \mathcal{A}, \mu)$. A subset G in S_1 is called bounded (resp., a.e. bounded) if G is so in $(S_1, \sigma(S_1, S_2))$, similarly we can define boundedness and a.e. boundedness for a subset of S_2. Let $\mathcal{B}$ be a family of a.e. bounded sets in S_2, for each F in $\mathcal{B}$ define $\mathcal{X}^F: S_1 \to L^+(\mu)$ be $X_p^F = \vee\{|\langle p, q\rangle| \mid q \in F\}$ for all p in S_1, denote $\{\mathcal{X}^F \mid F \in \mathcal{B}\}$ by $T_{\mathcal{B}}$, then $(S_1, T_{\mathcal{B}})$ is an E-RSN module over K with base $(\Omega, \mathcal{A}, \mu)$ if $\cup\mathcal{B}$ separates points in S_1, namely $\langle p_1, q\rangle = \langle p_2, q\rangle$ for all q in $\cup\mathcal{B}$ implies $p_1 = p_2$, we always assume $\mathcal{B}$ has this separation property. $T_{\mathcal{B}}$ is called admissible if $(S_1, T_{\mathcal{B}})^* \supset S_2$. If $\{\mathcal{X}^d\}_{d\in\Gamma}$ is a family of mappings from S_1 to $L^+(\mu)$ such that $(S_1, \{\mathcal{X}^d\}_{d\in\Gamma})$ is an E-RSN module over K with base $(\Omega, \mathcal{A}, \mu)$ and such that $(S_1, \{\mathcal{X}^d\}_{d\in\Gamma})^* = S_2$, then $\{\mathcal{X}^d\}_{d\in\Gamma}$ is called a consistent E-random seminorm module structure on S_1 with respect to $\langle S_1, S_2\rangle$ (briefly., a consistent E-$RSNM$-structure on S_1).

Based on Theorem 7.8 and Definition 7.6, we can further discuss the deep and complicated topics corresponding to consistent and admissible topologies of traditional duality theory (e.g., see [68]).

As already mentioned in Section 4, the notion of "RN spaces" employed in our initial work essentially amounts to the current notion of E-RN spaces. Based on the author's work [31], we further presented and studied the so called "strong E-RN spaces" in [35, 41, 42, 32] in order to establish such relations between continuity and a.e. boundedness of an E-random linear functional defined on these strong spaces as are described in Theorem 4.1. Such strong E-RN spaces and E-RIP spaces were also employed in [40, 66, 69]. However, as pointed out in [44, 34], although strong E-RN spaces possess more additional algebraic structures than general E-RN spaces, they are too weak to guarantee Theorem 4.1 to hold. Thus the very E-RN modules and E-RSN modules play an essential role in the deep development of E-random conjugate spaces, and what is more, almost all substantial results center on E-RN modules and E-RSN modules as illustrated in Sections 5, 6 and 7. Just motivated by the above observation, the author directly introduced in [52] the notion of an E-random duality as in Definition 7.4.

Nearly written at the same time as my paper [52], a recent paper [69] by You and Zhu also independently gave the notion of an E-random duality $\langle S_1, S_2\rangle$ over K with base $(\Omega, \mathcal{A}, \mu)$ (a probability space) where S_1 and S_2 are only linear spaces over K and $\langle\cdot, \cdot\rangle: S_1\times S_2 \to L(\mu, K)$ is only a bilinear mapping, which is much weaker than Definition

7.4, hence did not yield any substantial results in [69]. In [69] they also gave the notion of an a.e. bounded E-random linear functional f on an E-$RPNL$ space $(S, \{\mathscr{X}^d\}_{d\in\Gamma})$ over K with base $(\Omega, \mathscr{A}, \mu)$ as follows: f is called a.e. bounded if there exist some d in Γ and ξ in $L^+(\mu)$ such that $|f(p)| \leq \xi \cdot X_p^d$ for all p in S, this definition is both a special case of that given in [32, 34] (see Remark 7.5) and too restrictive in the sense that even if $(\Omega, \mathscr{A}, \mu)$ is trivial a continuous linear functional need not be a.e. bounded according to their definition. Thus the notion of an E-random conjugate space derived from such an a.e. bounded E-random linear functional makes them in [69] only discuss the (ϵ, λ)-linear topologies and not go further.

8 Random Metric Theory and Functional Analysis

E-RN modules, E-RIP modules and E-RSN modules are a natural generalization of ordinary normed spaces, inner product spaces and locally convex spaces (these prototypes amount to the case when the base space $(\Omega, \mathscr{A}, \mu)$ in question is trivial, i.e., $\mathscr{A} = \{\Omega, \emptyset\}$ and $\mu(\Omega) = 1$). However, when $(\Omega, \mathscr{A}, \mu)$ is nontrivial, in partivular when $\mathscr{A}$ contains no atoms, they are consideraby distinct from their prototypes at least in that 1) the latter are always locally convex linear topological spaces, and thus on which there always exist sufficiently many nontrivial continuous (or, bounded) linear functionals, whereas the former are, generally, a class of locally nonconvex topological modules over the locally noncnvex topological algebra $L(\mu, K)$, on which there exists not even a nontrivial continuous linear functional as was stated in our early joint paper [70] as far as E-RN spaces are concerned, and thus the theory of classical conjugate spaces fails to serve them, at this case what play crucial roles is the theory of E-random conjugate spaces; 2) the former possess more complicated and richer stratification structures than the latter, e.g., let $(S, \mathscr{X})$ be an E-RN module with base $(\Omega, \mathscr{A}, \mu)$ and p be an arbitrary element of S, then S contains every A-stratification $\tilde{I}_A \cdot p$ of p for each $A \in \mathscr{A}$. Sections 5, 6 and 7 have played an essential role in the deep developments of the theories of random conjugate spaces and random dualities, while it is such stratification structures that lead more complicated situations to happen frequently in the course of the studies of random metric theory.

The following observation merits speculation on methodology. Let $(S, \mathscr{X})$ be an E-RN module over K with a probability space $(\Omega, \mathscr{A}, \mu)$ as base, such an object as $(S, \mathscr{X})$ has, of course, been considered in both probability theory and functional analysis, for example, ordinarily $(S, \mathscr{X})$ is regarded as a quasinormed linear space $(S, \|\cdot\|)$ where $\|\cdot\|: S \to [0, +\infty)$ is defined by $\|p\| = \int_\Omega (X_p \wedge 1)\,d\mu$ (or, $\int_\Omega \frac{X_p}{1+X_p}\,d\mu$) for all p in S. It is fairly well known that $\|\cdot\|$ doesn't possess the homogeneity property and that the triangle inequality "$\|p+q\| \leq \|p\| + \|q\|$" is only a weaker property derived from a quite strong triangle inequality "$X_{p+q} \leq X_p + X_q$". Suffice it to say that it is the RN-module structure that captures the nature of $(S, \mathscr{X})$. Most important is that there exist many natural connections between E-RN modules and normed spaces, from which we can have a new and deeper insight into classical space theories, e. g., $(L^p(S))'$ can be unifiedly and elegantly represented as $L^q(S^*)$, and so on.

9 Random Metric Theory and the Theory of Probabilistic Metric Spaces

Random metric theory originated from the theory of PM spaces (e. g., see [15]), it has undergone a systematic and deep development in China in connection with both functional analysis and random functional analysis. Returning to the theory of PM spaces, there is no doubt that these developments of random metric theory are one of the most substantial contributions to the theory of PM spaces since RM spaces and RN spaces of the E-type or F-type just determine an extensive class of PM spaces and PN spaces in a natural way, respectively. In particular, the development of the theory of random conjugate spaces further induces a speculation-invited problem: whether can we have a theory of probabilistic conjugate spaces suitable for the further development of general PN spaces?

RN spaces, in particular RN modules are endowed with considerably more structure than general PN spaces, e. g., the former possess the stronger triangle inequality and more additional measure-theoretic structure that yield the smooth definition and deep development of a. e. bounded random linear functionals. What is interesting is that the triangle inequality is unique for the former while the triangle inequality is various for the latter. It is the variety of the trianle inequalities that leads to the fascinating theories of t-norms and copulas, which lay bare the substance of complicated dependences of random variables, see [7, 8] for details.

It should be also pointed out that the distinction between working with distribution functions as in the theory of PM spaces and working with random variables or μ-measurable functions as in random metric theory is intrinsic and not just a matter of taste as illustrated in [6].

10 A Further Analysis of the Space-Randomized Approach to Random Functional Analysis

In the ordinary studies of general random operators, e. g., in the studies of random fixed point theorems and random solutions of random operator equations, either by the approximating approach (e. g., in [27] and the study of random operators of the contraction-type[26, 28, 29]) or by combining measurable selection theorems[55] and classical theories in functional analysis one can get a measurable object[29, 43, 54] under the condition of separability of the spaces in question. Just as already mentioned frequently in [32, 33], such a model to developing random functional analysis (called the operator-randomized approach to random functional analysis in [32, 33]) not only has not fully utilized the powerful tool provided by classical functional analysis yet, for example, many elegant and deep classical theorems valid originally for those spaces that need not be assumed separable are employed in random functional analysis frequently under the hypothesis on separability although separability is sometimes essential, but also in particular would considerably reduce a taste of classical space theory. It seems to us that random metric theory would complement the operator-randomized approach.

As Section 3 shows, with random elements smoothly put into the basic frameworks in F-random metric theory (also in E-random metric theory in the form of quotient spaces if permissible as in Sections 5 and 6), random operators can also be unifiedly studied under these frameworks. This not only would not cause any loss of random elements and random operators but also can provide new means to study them. The substance of the space-randomized approach to random functional analysis is regarding random functional analysis as an analysis based on random metric theory (both F-random metric theory and E-random metric theory), since random metric theory is an organic combination of measure theory (or, probability theory) and functional analysis, each deep development of random metric theory leads naturally to the simultaneous applications of random metric theory to random operator theory and functional analysis, which is characteristic of random metric theory and in turn the space-randomized approach, and in particular all the results of random approximations and random operators in nonseparable spaces obtained in Sections 3, 4, 5 and 6 by random metric theoretic method can not be obtained from measurable selection theorems currently available!

In view of possible applications of random metric theory to stochastic analysis, we should also have such frameworks as adapted, predictable or optionable F-random metric space processes, F-random normed space processes, etc, not merely the framework of E-space processes as already introduced in [5]. One will easily think of these matters whenever he takes a look at the definition of E-space processes and the book [58] by A. V. Skorohod. There is no doubt that task would be more hard but more profound!

Acknowledgment In concluding this paper I want to thank Professor Yang Ming-zhu, editor of this journal, for encouraging me to write this paper and for giving me and my colleagues the opportunity to make our work known to a larger circle of scholars. I also want to thank Professor Liu Ying-ming for giving the lectures on "the pointed approach to fuzzy topology" at Sichuan University in 1987 that slightly later considerably stimulated me to come to the idea of the space-randomized approach to random functional analysis and also for warmly providing their stimulating paper [71] for me in 1992. I also want to thank Professors Chang Kung-ching, Yan Shi-jian, Sun Shun-hua, Wu Cong-xin, Lin Qun and Ma Zhi-ming for giving me some valueable suggestions and constructive criticisms on the related topics in functional analysis and probability theory. In particular, I also want to thank Professor B. Schweizer for giving me warm support and considerable encouragement in the course of my work and also for providing much literature for me, and Professors A. N. Serstnev and F. F. Sultanbekov for kindly sending me the references [62, 63] in Feb. 2001. I also want to thank Professors You Zhao-yong and Lin Xi, my teachers as well as my colleagues for giving me much help in the course of my work, and my colleagues-Professors Zhu Lin-hu and Gong Fu-zhou for the pleasure I have had working with the four colleagues over the years. I also want to thank my colleagues at Xiamen Unuversity—Professors Zhang Fu-ji, Zhong Tong-de, Zheng Yao-hui, Lin Liang-yu, Cheng Li-xin and Lin Ya-nan for giving me much help for the last six years. Finally, support from Foundation for University key Teacher by the Ministry of Education, the National Natural Science Foundation of China (No. 10071063) and the Natural Science Foundation of Fujian Province of China (F99027) is gratefully acknowledged.

References:

[1] Menger K. Statistical metrics[J]. Proc Nat Acad Sci USA, 1942, 28: 535—537.
[2] Wald A. On a statistical generalization of metric spaces[J]. Proc Nat Acad Sci USA, 1943, 29: 196—197.
[3] Šchweizer B, Sklar A. Espaces m triques al atoires[J]. C R Acad Sci Paris, 1958, 247: 2092—2094.
[4] Serstnev A N. On a probabilistic generalization of metric spaces[J]. Kazan Gos Univ Ucen Zap, 1964, 124: 3—11.
[5] Schweizer B, Sklar A. Probabilistic Metric Spaces[M]. Elsevier-North Holland, New York, 1983.
[6] Schweizer B, Sklar A. Operations on distribution functions not derivable from operations on random variables[J]. Studia Math, 1974, 52: 43—52.
[7] Schweizer B, Wolff E F. On nonparametric measures of dependence for random variables[J]. Ann Statist, 1981, 9: 879—885.
[8] Schweizer B. Thirty years of copulas. In Advances in Probability Distribution Functions with Given Marginals: Beyond the Copulas[A]. G Dall-Aglio, S Kotz and G Salinetti. Mathematics and Its Applications[C]. Kluwer, Dordrecht, 1991, 67: 13—51.
[9] Darsow W F, Nguyen B, Olsen E T. Copulas and Markov processes[J]. Ill J Math, 1992, 36: 600—642.
[10] Nelsen R B. An Introduction to Copulas[M]. Lecture Notes in Statistics, V 139 Springer, New York, 1999.
[11] Schweizer B, Sklar A. Measures al atoires de I 'information[J]. C R Acad Sci Paris, 1969, 269A: 149—152.
[12] Janowitz M F, Schweizer B. Ordinal and percentile clustering[J]. Math Social Sciences, 1989, 18: 135—186.
[13] Schweizer B, Smital J. Measures of chaos and a spectral decomposition of dynamical systems on the interval[J]. Trans Amer Math Soc, 1994, 344: 737—754.
[14] Schweizer B. On the genesis of the notion of distributional chaos[J]. Rendiconti del seminario Matematico e Fisico di Milano, 1996, 66: 159—167.
[15] Špaček A. Note on K. Menger's probabilistic geometry[J]. Czechoslovak Math J, 1956, 81(6): 72—74.
[16] Spacek A. Random Metric Spaces[A]. Trans Second Prague Conf Information Theory, Decision Functions and Random Processes[C], 1960, 627—638. Academic Press, New York.
[17] Stevens R R. Metrically generated probabilistic metric spaces[J]. Fund Math, 1968, 61: 259—269.
[18] Sherwood H. On E-spaces and their relation to other classes of probabilistic metric spaces[J]. J London Math Soc, 1969, 44: 441—448.
[19] Sherwood H. Isomorphically isometric probabilistic normed linear spaces[J]. Stochastica, 1979, 3(2): 71—79.
[20] Mustari D H. On almost sure convergence in linear spaces of random variables[J]. Jheory Prob Appl, 1970, 15: 337—342.
[21] LIN Xi. Fixed Point Theorems in Probabilistic Metric and Probabilistic Normed Spaces and Applications[D]. Xi' an Jiaotong Univerity, 1981.
[22] YAN Jia-an. Martingales and Stochastic Integrals[M]. Shanghai: Shanghai Science and Technology Press, 1981.
[23] ZHU Lin-hu. Metric properties of Probabilistic Metric Spaces and Applications[D]. Xi'an Jiaotong University, 1988.
[24] YOU Zhao-yong, ZHU Lin-hu. Ekeland's variational principle on E-spaces[J]. J of Engg Math (China), 1988, 3: 1—7. (In Chinese).
[25] Hans O. Generalized random variables[A]. Jaroslav Kozesnik. Information Theory Statistical Decision Functions

Random Processes[C]. Publishing House of the CZECHOSLOVAK ACADEMY OF SCIENCES, 1957. 61—63.
[26] Hanš O. Random Operator Equations[A]. Proc 4th Berkeley Symp on Math Statist and Probability (1960)[C]. Univ California Press, Berkeley, California, 1961, 2: 185—202.
[27] Hanš O. Measurability of extensions of continuous random transforms[J]. Annals of Math Statist, 1959, 1152—1157.
[28] WANG Zi-kun. An introduction to random functional analysis[J]. Advances in Mathematics (China), 1962, 5(1): 45—71.
[29] Bharucha-Reid A T. Fixed point theorems in probabilistic analysis[J]. Bull Amer Math Soc, 1976, 82: 641—657.
[30] Neveu J. Mathematical Foundations of the Calculus of Probabilities[M]. Holden-Day, San Francisco, 1965.
[31] GUO Tie-xin. The Theory of Probabilistic Metric Spaces with Applications to Random Functional Analysis[D]. Xi'an Jiaotong University, 1989.
[32] GUO Tie-xin. Random Metric Theory and its Applications[D]. Xi'an Jiaotong University, 1992.
[33] GUO Tie-xin. A new approach to random functional analysis[A]. Proceedings of the first China postdoctoral academic congress (Ⅱ)[C]. China National Defence and Industry Press, 1993, 1: 1150—1154. (In Chinese)
[34] GUO Tie-xin. Module homomorphisms on random normed modules[J]. Chinese Northeastern Math J, 1996, 12(1): 102—114.
[35] GONG Fu-zhou. Probabilistic Metric Spaces with Applications to Stochastic Analysis[D]. Northwestern University (China), 1989.
[36] LIN Xi, GUO Tie-xin. Random inner product spaces[J]. Chinese Science Bulletin, 1990, 35(22): 1707—1709. (In Chinese).
[37] LIN Xi, GUO Tie-xin. Random metric spaces and their applications[J]. J of Math Research and Exposition (China), 1992, 12(4): 499—504.
[38] YOU Zhao-yong, GUO Tie-xin, GONG Fu-zhou. Random linear topological spaces[J]. J of Xi'an Jiaotong Univ, 1991, 25(6): 331—334. (Abstract in Chinese).
[39] LIN Xi, LI Chuan-mu. Orthoprojection theorems in random inner product spaces[J]. J of Engineering Mathematics (China), 1991, 8(2): 134—138. (In Chinese).
[40] LIU Qing-rong, GONG Fu-zhou. The orthodecomposition theorem in class of random inner product spaces and its applications[J]. Chinese Annals of Mathematics (Series A), 1992, 13(3): 296—305. (In Chinese).
[41] YOU Zhao-yong, ZHU Lin-hu, GUO Tie-xin. Random conjugate spaces of a class of quasinormed linear spaces [J]. J of Xi'an jiaotong University, 1991, 25(3): 133—134. (Abstract in Chinese).
[42] GONG Fu-zhou. Random conjugate spaces of a class of quasinormed linear spaces and completeness of the space of all almost surely bounded linear operators on these spaces[J]. J of Northwestern University (China) (Natural Science Edition), 1993, 5: 404—408. (In Chinese).
[43] Diestel J, Uhl Jr J J. Vector Measures[M]. Math Surveys, No 15, Amer Math Soc Providence, Rhodo Island, 1977.
[44] GUO Tie-xin. Extension theorems of continuous random linear operators on random domains[J]. J Math Anal Appl, 1995, 193(1): 15—27.
[45] GUO Tie-xin, YOU Zhao-yong. Riesz representation theorems in complete random inner product modules[J]. Chinese Annals of Mathematics (Ser. A), 1996, 17(3): 361—364. (In Chinese).
[46] GUO Tie-xin. Representation theorems of the dual of Lebesgue-Bochner function spaces[J]. Science in China (Ser. A), 2000, 43(3): 234—243.
[47] GUO Tie-xin. The Radon-Nikodým property of conjugate spaces and the w^*-equivalence theorem for w^*-measurable functions[J]. Science in China (Ser. A), 1996, 39(10): 1034—1041.
[48] GUO Tie-xin. Characterizations for a complete random normed modules to be random reflexive[J]. J of Xiamen University (Natural Science Edition), 1997, 36(4): 499—502. (In Chinese).
[49] GUO Tie-xin. Some basic theories of random normed linear spaces and random inner product spaces[J]. Acta Analysis Functionalise Applicatae, 1999, 1(2): 160—184.
[50] YOU Zhao-yong, GUO Tie-xin. Pointwise best approximation in the space of strongly measurable functions with applications to best approximation in $L^p(\mu, X)$[J]. J of Approximation Theory, 1994, 78(3): 314—320.
[51] GUO Tie-xin, YOU Zhao-yong. A note on pointwise best approximation[J]. J of Approximation Theory, 1998, 93(2): 344—347.

[52] GUO Tie-xin. Random dual spaces[J]. J of Xiamen University(Natural Science Edition), 1997, 36(2): 167—170. (In Chinese).
[53] Engl H W. A general stochastic fixed-point theorem for continuous random operators on stochastic domains[J]. J Math Anal Appl, 1978, 66: 220—231.
[54] Engl H W. Random fixed point theorems for multivalued mappings[J]. Pacific J of Mathematics, 1978, 76(2): 351—360.
[55] Wagner D H. Survey of measurable selection theorems[J]. SIAM J Control and Optimization, 1977, 15(5): 859—903.
[56] Mourier E. Éléments aléatoires dans un espace de Banach[J]. Annales de l'Institut Henri Poincaré, 1953, 160—244.
[57] Dunford N, Schwartz J T. Linear Operators (Ⅰ)[M]. London: Interscience, 1957.
[58] Skorohod A V. Random Linear Operators[M]. D Reidel Comp, Holland, 1984.
[59] Itô Kiyosi, Nawata Masako. Regularization of linear random functionals[M]. Lecture Notes in Mathematics, 1982, 1021: 257—267.
[60] Phelps R R. Convex Functions, Differentiability and Monotone Operators[M]. Lecture Notes in Mathematics, Vol. 364, Springer-Verlag, 1989.
[61] Dudley R M, Philipp W. Invariance principles for sums of Banach space valued random elements and empirical processes[J]. Z Wahr Verw Geb, 1983, 62: 509—552.
[62] Sultanbekov F F. On random functionals in spaces of strongly measurable functions[J]. Issled Priklad Mat, Izd-vo KGU, 1979, 6: 74—82.
[63] Sultanbekov F F. On weak random functionals in spaces of strongly measurable functions[J]. Konstruktivnaja Teor Funkcij i Funkcionalnij Analiz, Izd-vo KGU, 1979, 2: 88—92.
[64] Ionescu Tulcea A, Ionescu Tulcea C. On the lifting property (Ⅰ)[J]. J Math Anal Appl, 1961, 3: 537—546.
[65] Ionescu Tulcea A, Ionescu Tulcea C. On the lifting property (Ⅱ)[J]. J Math Mech, 1962, 11(5): 773—795.
[66] GONG Fu-zhou. The spectral decomposition theorems for certain class of unbounded linear random operator in a class of complete quasi-normed linear spaces[J]. Advances in Mathematics (China), 1994, 23(5): 432—438. (In Chinese).
[67] GUO Tie-xin. A weak equivalence theorem for weak random elements with values in weakly compactly generated Banach spaces and its applictions[J]. Chinese Science Bulletin, 1996, 40(1): 6—10.
[68] Schaefer H H. Topological Vector Spaces[M]. World Publishing Corporation, Springer-Verlag, 1971.
[69] YOU Zhao-yong, ZHU Lin-hu. Random conjugate spaces and random duals[J]. Journal of Engineering Mathematics (China), 1996, 13: 77—89.
[70] GUO Tie-xin, ZHU Lin-hu. Random metric space system[J]. J of Engineering Mathematics(China), 1991, 8(2): 208—212. (In Chinese).
[71] LIU Ying-ming, LUO Mao-kang. Lattice-valued Hahn-Dieudonné-Tong insertion theorem and stratification structure[J]. Topology and its applications, 1992, 45: 173—188.
[72] Sherwood H. Complete probabilistic metric spaces[J]. Z Wahr Verw Geb, 1971, 20: 117—128.

随机度量理论及其应用在我国最近进展的综述

郭铁信

(厦门大学数学系, 福建 厦门 361005)

摘要: 本文旨在全面综述随机度量理论及其应用过去十年在我国发展过程中所获得的主要结果与思想方法. 全文由十节组成, 第一节对我们工作的背景——概率度量空间与随机度量空间理论作一简单的介绍; 第二节给出某些有关随机泛函分析及取值于抽象空间的可测函数的预备知识; 第三节阐明随机泛函分析与原始随机度量理论(本文称之为 F-随机度量理论)的整体关系; 主要结果是在随机元生成空间上给出自然且合

理的随机度量与随机范数的构造, 从而将随机元与随机算子理论的研究纳入随机度量理论框架; 主要思想是将随机泛函分析视为随机度量空间体系上的分析学而统一地发展, 从而形成了发展随机泛函分析的一个新的途径—— 空间随机化途径; 除此之外, 在本节我们也从随机过程理论的观点出发首次提出对应于随机度量理论原始版本的一种新的随机共轭空间理论(叫作 F-随机共轭空间理论) , 它的突出优点是能保持象随机过程的样本性质这样更精细的特性(本节由作者的工作构成) ; 在第四节, 基于作者最近提出的随机度量理论的一个新的版本(本文称之为 E-随机度量理论) , 从传统泛函分析的角度对过去已被发展起来的随机共轭空间理论(本文称之为 E-随机共轭空间理论) 的基本结果进行系统整理并给以全新的处理(本节内容整体上由作者最近的一篇论文构成, 也尤其提到朱林户等人的重要工作) ; 在本节我们也以相当的篇幅论述 F-随机共轭空间理论与 E-随机共轭空间理论的内在关系与本质差异. 在下面紧跟的两节, 致力于 E-随机共轭空间理论深层次的结果, 尤其突出了 E-随机赋范模与传统的赋范空间、E-随机共轭空间与经典共轭空间之间的内在联系; 在第五节给出了几类 E-随机赋范模的 E-随机共轭空间的表示定理(主要由作者的工作, 作者与游兆永及林熙合作的工作, 还有巩馥州与刘清荣合作的工作组成) ; 在第六节给出完备 E-随机赋范模为随机自反的特征化定理(主要由作者及合作者的工作组成) . 尤其在第五及第六节中, 我们给出随机度量理论在随机泛函分析及经典 Banach 空间中若干实质性的应用; 第七节简要给出 E-随机赋半范模及 E-随机对偶系理论初步; 第八节简单阐明随机度量理论与泛函分析的关系; 第九节简单阐明了随机度量理论与概率度量空间理论的关系. 最后在第十节结合随机度量理论, Banach 空间理论及随机泛函分析对发展随机泛函分析的空间随机化途径的合理性与优越性作了进一步的分析.

关键词: 概率度量空间理论; 随机度量理论; 随机泛函分析; 空间随机化途径; 随机赋范模; 随机内积模; 随机共轭空间; 随机对偶系