第 39 卷 第 4 期　　厦门大学学报(自然科学版)　　Vol. 39 No. 4
2000 年 7 月　　Journal of Xiamen University (Natural Science)　　Jul. 2000

·研究简报·

文章编号:0438-0479(2000)04-0560-06

可控增长条件下一类椭圆型方程弱解的局部极值原理

韩 丕 功

(厦门大学数学系,福建 厦门 361005)

摘要:研究二阶非线性椭圆型偏微分方程——$-\mathrm{div}A(x,u,\nabla u)+B(x,u,\nabla u)=\mu$,在可控增长结构条件——$A(x,z,\eta)\cdot\eta\geqslant\lambda|\eta|^p-\Lambda|\eta|^{p^*}-1$,$|A(x,z,\eta)|\leqslant\Lambda_1(|\eta|^{p-1}+|z|^{p^*(1-\frac{1}{p})}+1)$,$|B(x,z,\eta)|\leqslant\Lambda(|\eta|^{p(1-\frac{1}{p^*})}+|z|^{p^*-1}+1)$ 下,应用 Moser 迭代法得出弱解的局部极值原理,并进一步得出弱解的内部估计和全局估计.

关键词:可控增长条件;椭圆型方程;弱解

中图分类号:O 157.29　　　　**文献标识码**:A

1 定义及主要结果

考虑散度型方程:

$$-\mathrm{div}A(x,u,\nabla u)+B(x,u,\nabla u)=\mu \tag{1}$$

其中 $A(x,z,\eta)$,$B(x,z,\eta)$ 分别是定义在 $\Omega\times R^1\times R^N$ 上,关于 x 可测,z,η 连续的向量函数和实函数,μ 属于广义空间 $M_{\lambda_1}^{-1,q}$ 中的函数.这里 $M_{\lambda_1}^{-1,q}=\{T\in W^{-1,q}(\Omega);\sup\limits_{\rho>0}\rho^{-\frac{\lambda_1}{q}}\|T\|_{W^{-1,q}(\Omega\cap B_\rho)}<+\infty\}$.$\Omega$ 为 R^N 中有界开集,$B_\rho=\{y\in R^N;|y-x|<\rho,x\in\Omega\}$,$\lambda_1>N$ 为正常数,q 的规定见下面,假定任意方程(1) 满足结构条件:对于任意 $(x,z,\eta)\in\Omega\times R^1\times R^N$

$$A(x,z,\eta)\cdot\eta\geqslant\lambda|\eta|^p-\Lambda|z|^{p^*}-1 \tag{2}$$

$$|A(x,z,\eta)|\leqslant\Lambda_1|\eta|^{p-1}+|z|^{p^*(1-\frac{1}{p})}+1) \tag{3}$$

$$|B(x,z,\eta)|\leqslant\Lambda(|\eta|^{p(1-\frac{1}{p^*})}+|z|^{p^*-1}+1) \tag{4}$$

其中 $\lambda,\Lambda_1,\Lambda$ 均为正常数,$1<p<N$,$p^*=\dfrac{Np}{N-p}$ 且 $\dfrac{1}{p}+\dfrac{1}{q}$.条件(2) ~ (4) 称为可控增长条件.

定义 1 称函数 $u\in W_{loc}^{1,p}(\Omega)$ 为方程(1) 的弱解,如果对于任意 $\varphi\in C_0^1(\Omega)$,都有

收稿日期:1999-07-15

作者简介:韩丕功 (1974—),男,硕士生.

$$\int_\Omega A(x,u,\nabla u)\nabla\varphi \mathrm{d}x + \int_\Omega B(x,u,\nabla u)\varphi \mathrm{d}x = \int_\Omega \varphi \mathrm{d}\mu \tag{5}$$

定理 1　设方程(1) 满足结构条件(2) ～ (4),又设 $u \in W^{1,p}_{loc}(\Omega)$ 是方程(1) 的非负弱解,则对任意球 $B_R \subset\subset \Omega$ 及任意 $0 < \theta < 1$,有

$$\operatorname{ess\,sup}_{B_R} u \leqslant C(1 + (\fint_{B_R} u^p \mathrm{d}x)^{\frac{1}{p}}) \tag{6}$$

其中 C 仅依赖于 $N,\lambda,\lambda_1,\Lambda_1,\Lambda,P,\theta,\mathrm{diam}\Omega$. $\fint_{B_R} u^p \mathrm{d}x = \frac{1}{|B_R|}\int_{B_R} u^p \mathrm{d}x$.

2　定理 1 的的证明

证　设 u 是方程(1) 的非负弱解,在式(5) 中,取 $\varphi = \xi^p \bar{u}^\beta$,其中 $\xi \in C_0^\infty(B_R)$ 且 $0 \leqslant \xi \leqslant 1$;$\bar{u} = u + 1$,$\beta \geqslant 1$ 待定,则有

$$\beta\int_\Omega A(x,u,\nabla u)\cdot\nabla\bar{u}\xi^p\bar{u}^{\beta-1}\mathrm{d}x = -p\int_\Omega A(x,u,\nabla u)\xi^{p-1}\nabla\xi\bar{u}^\beta \mathrm{d}x -$$
$$\int_\Omega B(x,u,\nabla u)\xi^p\bar{u}^\beta \mathrm{d}x + \int_\Omega \xi^p\bar{u}^\beta d\mu \tag{7}$$

利用结构条件(2) ～ (4) 有

$$\beta\int_\Omega A(x,u,\nabla u)\cdot\nabla\bar{u}\xi^p\bar{u}^{\beta-1}\mathrm{d}x \geqslant \beta\int_\Omega \xi^p\bar{u}^{\beta-1}(\lambda|\nabla\bar{u}|^p - \Lambda\bar{u}^{p^*} - 1)\mathrm{d}x$$

$$|-p\int_\Omega \xi^{p-1}\bar{u}^\beta A(x,u,\nabla u)\cdot\nabla\xi \mathrm{d}x| \leqslant p\Lambda_1\int_\Omega \xi^{p-1}\bar{u}^\beta|\nabla\xi|(|\nabla\bar{u}|^{p-1} + \bar{u}^{p^*(1-\frac{1}{p})} + 1)\mathrm{d}x$$
$$\leqslant p\Lambda_1\int_\Omega \xi^{p-1}\bar{u}^\beta|\nabla\xi|\cdot|\nabla\bar{u}|^{p-1}\mathrm{d}x + p\Lambda_1\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^{\beta+p^*(1-\frac{1}{p})}\mathrm{d}x +$$
$$p\Lambda_1\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^\beta \mathrm{d}x$$

$$|\int_\Omega B(x,u,\nabla u)\xi^p\bar{u}^\beta \mathrm{d}x| \leqslant \Lambda\int_\Omega(|\nabla\bar{u}|^{p(1-\frac{1}{p^*})} + \bar{u}^{p^*-1} + 1)\xi^p\bar{u}^\beta \mathrm{d}x \leqslant$$
$$\Lambda\int_\Omega \xi^p\bar{u}^\beta|\nabla\bar{u}|^{p(1-\frac{1}{p^*})}\mathrm{d}x + \Lambda\int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x + \Lambda\int_\Omega \xi^p\bar{u}^\beta \mathrm{d}x$$

将上述这些估计式代入式(7) 中,并注意到 $\beta \geqslant 1$,得

$$\lambda\int_\Omega \xi^p\bar{u}^{\beta-1}|\nabla\bar{u}|^p \mathrm{d}x \leqslant \Lambda\int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x + \int_\Omega \xi^p\bar{u}^{\beta-1}\mathrm{d}x + p\Lambda_1\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^\beta|\nabla\bar{u}|^{p-1}\mathrm{d}x +$$
$$p\Lambda_1\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^{\beta+p^*(1-\frac{1}{p})}\mathrm{d}x + p\Lambda_1\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^\beta \mathrm{d}x + \Lambda\int_\Omega|\nabla\bar{u}|^{p(1-\frac{1}{p^*})}\xi^p\bar{u}^\beta \mathrm{d}x +$$
$$\Lambda\int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x + \Lambda\int_\Omega \xi^p\bar{u}^\beta \mathrm{d}x + \int_\Omega \xi^p\bar{u}^\beta \mathrm{d}\mu \tag{8}$$

下面对式(8) 中右端每一项进行估计,利用 Young 不等式,有

$$\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^\beta|\nabla\bar{u}|^{p-1}\mathrm{d}x \leqslant \varepsilon\int_\Omega \xi^p\bar{u}^{\beta-1}|\nabla\bar{u}|^p \mathrm{d}x + c(\varepsilon)\int_\Omega|\nabla\xi|^p\bar{u}^{\beta+p-1}\mathrm{d}x$$

$$\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^{\beta+p^*(1-\frac{1}{p})}\mathrm{d}x \leqslant \int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x + \int_\Omega|\nabla\xi|^p\bar{u}^{\beta+p-1}\mathrm{d}x$$

$$\int_\Omega|\nabla\bar{u}|^{p(1-\frac{1}{p^*})}\xi^p\bar{u}^\beta \mathrm{d}x \leqslant \varepsilon\int_\Omega \xi^p\bar{u}^{\beta-1}|\nabla\bar{u}|^p \mathrm{d}x + c(\varepsilon)\int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x$$

$$\int_\Omega \xi^{p-1}|\nabla\xi|\bar{u}^\beta \mathrm{d}x \leqslant \int_\Omega(\frac{1}{q}\xi^{q(p-1)} + \frac{1}{p}|\nabla\xi|^p)\bar{u}^\beta \mathrm{d}x \leqslant \int_\Omega(\xi^p + |\nabla\xi|^p)\bar{u}^\beta \mathrm{d}x \leqslant$$

$$\int_\Omega (\xi^p + |\nabla\xi|^p)(\frac{\beta}{\beta+p-1}\bar{u}^{\beta+p-1} + \frac{P-1}{\beta+p-1})\mathrm{d}x \leqslant$$

$$\int_\Omega (\xi^p + |\nabla\xi|^p)\bar{u}^{\beta+p-1}\mathrm{d}x + \int_\Omega (\xi^p + |\nabla\xi|^p)\mathrm{d}x$$

$$\int_\Omega \xi^p\bar{u}^\beta \mathrm{d}x \leqslant \int_\Omega \xi^p(\frac{\beta}{\beta+p+1}\bar{u}^{\beta+p-1} + \frac{p-1}{\beta+p-1})\mathrm{d}x \leqslant \int_\Omega \xi^p\bar{u}^{\beta+p-1}\mathrm{d}x + \int_\Omega \xi^p \mathrm{d}x$$

$$\int_\Omega \xi^p\bar{u}^{\beta-1}\mathrm{d}x \leqslant \int_\Omega \xi^p(\frac{\beta-1}{\beta+p-1}\bar{u}^{\beta+p-1} + \frac{p}{\beta+p-1})\mathrm{d}x \leqslant \int_\Omega \xi^p\bar{u}^{\beta+p-1}\mathrm{d}x + \int_\Omega \xi^p \mathrm{d}x$$

$$|\int_\Omega \xi^p\bar{u}^\beta \mathrm{d}\mu| = |<\mu, \xi^p\bar{u}^\beta>| \leqslant \|\mu\|_{M_{\lambda_1}^{-1,q}} \|\nabla(\xi^p\bar{u}^\beta)\|_{L^p(\Omega)} \leqslant CR^{\frac{\lambda_1}{q}} \|\nabla(\xi^p\bar{u}^\beta)\|_{L^p\Omega} \leqslant$$

$$CR^{\lambda_1} + \int_\Omega |p\xi^{p-1}(\nabla\xi)\cdot\bar{u}^\beta + \beta\xi^p\bar{u}^{\beta-1}\nabla\bar{u}|^p\mathrm{d}x \leqslant$$

$$CR^{\lambda_1} + (2p)^p\int_\Omega \xi^{p(p-1)}\bar{u}^{\beta p}|\nabla\xi|^p\mathrm{d}x + (2\beta)^p\int_\Omega \xi^{p2}\bar{u}^{(\beta-1)p}|\nabla\bar{u}|^p\mathrm{d}x$$

将上这些估计式代入式(8)中得

$$\int_\Omega \xi^p\bar{u}^{\beta-1}|\nabla\bar{u}|^p\mathrm{d}x \leqslant C(\int_\Omega \xi^p\bar{u}^{\beta+p^*-1}\mathrm{d}x + \int_\Omega \xi^{p(p-1)}\bar{u}^{\beta p}|\nabla\xi|^p\mathrm{d}x +$$
$$\beta^p\int_\Omega \xi^{p^2}\bar{u}^{(\beta-1)p}|\nabla\bar{u}|^p\mathrm{d}x + \int_\Omega (\xi^p + |\nabla\xi|^p)\bar{u}^{\beta+p-1}\mathrm{d}x +$$
$$\int_\Omega (\xi^p + |\nabla\xi|^p\mathrm{d}x + R^{\lambda_1}) \tag{9}$$

其中常数 C 与 β 无关. 令 $W = \bar{u}^{\frac{\beta+p-1}{p}}$,则式(9)写为

$$(\frac{p}{\beta+p-1})^p\int_\Omega \xi^p|\nabla W|^p\mathrm{d}x \leqslant C(\int_\Omega \xi^p W^{\frac{p(\beta+p^*-1)}{\beta+p-1}}\mathrm{d}x + \int_\Omega \xi^{p(p-1)}|\nabla\xi|^p W^{\frac{p^2\beta}{\beta+p-1}}\mathrm{d}x +$$
$$\int_\Omega (\xi^p + |\nabla\xi|^p)W^p\mathrm{d}x + \int_\Omega \xi^{p2}W^{\frac{p(\beta p-\beta-p+1)}{\beta+p-1}}|\nabla W|^p\mathrm{d}x +$$
$$\int_\Omega (\xi^p + |\nabla\xi|^p)\mathrm{d}x + R^{\lambda_1}) \tag{10}$$

记 $r = \beta + p - 1$,由式(10)得

$$\int_\Omega \xi^p|\nabla W|^p\mathrm{d}x \leqslant Cr^p(\int_\Omega \xi^p W^{\frac{p(\beta+p^*-1)}{\beta+p-1}}\mathrm{d}x + \int_\Omega \xi^{p(p-1)}|\nabla\xi|^p W^{\frac{p^2\beta}{\beta+p-1}}\mathrm{d}x +$$
$$\int_\Omega \xi^{p^2}W^{\frac{p(\beta p-\beta-p+1)}{\beta+p-1}}|\nabla W|^p\mathrm{d}x + \int_\Omega (\xi^p + |\nabla\xi|^p)W^p\mathrm{d}x +$$
$$\int_\Omega |\nabla\xi|^p\mathrm{d}x + R^{\lambda_1}) \tag{11}$$

而 $$\int_\Omega |\nabla(\xi W)|^p\mathrm{d}x = \int_\Omega |(\nabla\xi)\cdot W + \xi\cdot\nabla W|^p\mathrm{d}x \leqslant$$
$$2^{p-1}\int_\Omega |\nabla\xi|^p W^p\mathrm{d}x + 2^{p-1}\int_\Omega \xi^p|\nabla W|^p\mathrm{d}x \tag{12}$$

将式(11)代入式(12)中得

$$\int_\Omega |\nabla(\xi W)|^p\mathrm{d}x \leqslant Cr^p(\int_\Omega \xi^p W^{\frac{p(\beta+p^*-1)}{\beta+p-1}}\mathrm{d}x + \int_\Omega \xi^{p(p-1)}|\nabla\xi|^p W^{\frac{p^2\beta}{\beta+p-1}}\mathrm{d}x +$$
$$\int_\Omega \xi^{p^2}W^{\frac{p(\beta p-\beta-p+1)}{\beta+p-1}}|\nabla W|^p\mathrm{d}x + \int_\Omega (\xi^p + |\nabla\xi|^p)W^p\mathrm{d}x + \int_\Omega |\nabla\xi|^p\mathrm{d}x + R^{\lambda_1}) \tag{13}$$

下面估计式(13)中右端前3项. 引进记号:

$$\Omega_{1,M}^+ = \{x \in \Omega | W^{\frac{p(p^*-p)}{\beta+p-1}} > M\}; \qquad \Omega_{1,M}^- = \{x \in \Omega | W^{\frac{p(p^*-p)}{\beta+p-1}} \leqslant M\};$$

$$\Omega_{2,M}^{+}=\{x\in\Omega|W^{\frac{p(p-1)(\beta-1)}{\beta+p-1}}>M\};\qquad \Omega_{2,M}^{-}=\{x\in\Omega|W^{\frac{p(p-1)(\beta-1)}{\beta+p-1}}\leqslant M\};$$

$$\Omega_{3,M}^{+}=\{x\in\Omega|W^{\frac{p(\beta p-\beta-p)}{\beta+p-1}}|\nabla W|^{p}>M\};\qquad \Omega_{3,M}^{-}=\{x\in\Omega|W^{\frac{p(\beta p-\beta-p)}{\beta+p-1}}|\nabla W|^{p}\leqslant M\},$$

其中 M 为任一正实数.

$$\int\xi_{\Omega}^{p}W^{\frac{p(\beta+p^{*}-1)}{\beta+p-1}}\mathrm{d}x=\int_{\Omega_{1,M}^{+}}\xi^{p}W^{p}\cdot W^{\frac{p(p^{*}-p)}{\beta+p-1}}\mathrm{d}x+\int_{\Omega_{1,M}^{-}}\xi^{p}W^{p}\cdot W^{\frac{p(p^{*}-p)}{\beta+p-1}}\mathrm{d}x\leqslant$$

$$\Big(\int_{\Omega_{1,M}^{+}}|\xi W|^{p^{*}}\mathrm{d}x\Big)^{\frac{p}{p^{*}}}\cdot\Big(\int_{\Omega_{1,M}^{+}}W^{\frac{pp^{*}}{\beta+p-1}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}+M\int_{\Omega_{1,M}^{+}}\xi^{p}W^{p}\mathrm{d}x\leqslant$$

$$\Big(\int_{\Omega}|\xi W|^{p^{*}}\mathrm{d}x\Big)^{\frac{p}{p^{*}}}\cdot\Big(\int_{\Omega_{1,M}^{+}}W^{\frac{pp^{*}}{\beta+p-1}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}+M\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\leqslant$$

$$C_{0}\int_{\Omega}|\nabla(\xi W)|^{p}\mathrm{d}x\cdot\Big(\int_{\Omega_{1,M}^{+}}W^{\frac{pp^{*}}{\beta+p-1}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}+M\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\tag{14}$$

其中 C_0 为嵌入常数,当 M 充分大时,可使得

$$\Big(\int_{\Omega_{1,M}^{+}}W^{\frac{pp^{*}}{\beta+p-1}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}<\frac{1}{4C_{o}r^{p}}$$

将此式代入式(14)中得

$$\int_{\Omega}\xi^{p}W^{\frac{p(\beta+p^{*}-1)}{\beta+p-1}}\mathrm{d}x<\frac{1}{4r^{p}}\int_{\Omega}|\nabla(\xi W)|^{p}\mathrm{d}x+M\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\tag{15}$$

$$\int_{\Omega}\xi^{p(p-1)}|\nabla\xi|^{p}W^{\frac{p^{2}\beta}{\beta+p-1}}\mathrm{d}x\leqslant\sup_{\Omega}|\nabla\xi|^{p}\int_{\Omega_{2,M}^{+}}\xi^{p}W^{p}W^{\frac{p(p-1)(\beta-1)}{\beta+p-1}}\mathrm{d}x+$$

$$\sup_{\Omega}|\nabla\xi|^{p}\int_{\Omega_{2,M}^{-}}\xi^{p}W^{p}W^{\frac{p(p-1)(\beta-1)}{\beta+p-1}}\mathrm{d}x\leqslant$$

$$\sup_{\Omega}|\nabla\xi|^{p}\cdot\Big(\int_{\Omega_{2,M}^{+}}|\xi W|^{p^{*}}\mathrm{d}x\Big)^{\frac{p}{p^{*}}}\cdot\Big(\int_{\Omega_{2,M}^{+}}W^{\frac{p^{*}\cdot P(p-1)(\beta-1)}{(p^{*}-p)(\beta+p-1)}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}+$$

$$M\sup_{\Omega}|\nabla\xi|^{p}\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\leqslant C_{1}\cdot\sup_{\Omega}|\nabla\xi|^{p}\cdot\int_{\Omega}|\nabla(\xi W)|^{p}\mathrm{d}x\cdot$$

$$\Big(\int_{\Omega_{2,M}^{+}}W^{\frac{p^{*}p(p-1)(\beta-1)}{(p^{*}-p)(\beta+p-1)}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}+M\sup_{\Omega}|\nabla\xi|^{p}\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\tag{16}$$

其中 C_1 为嵌入常数,当 M 充分大(要求 $M>\sup\limits_{\Omega}|\nabla\xi|^{p}$)时,可使得

$$\Big(\int_{\Omega_{2,M}^{+}}W^{\frac{p^{*}p(p-1)(\beta-1)}{(p^{*}-p)(\beta+p-1)}}\mathrm{d}x\Big)^{1-\frac{p}{p^{*}}}<\frac{1}{4C_{1}r^{p}\cdot\sup\limits_{\Omega}|\nabla\xi|^{p}}$$

将此式代入式(16)中得

$$\int_{\Omega}\xi^{p(p-1)}|\nabla\xi|^{p}W^{\frac{p^{2}\beta}{\beta+p-1}}\mathrm{d}x<\frac{1}{4r^{p}}\int_{\Omega}|\nabla(\xi W)|^{p}\mathrm{d}x+M^{2}\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x\tag{17}$$

$$\int_{\Omega}\xi^{p^{2}}W^{\frac{p(\beta p-\beta-p+1)}{\beta+p-1}}|\nabla W|^{p}\mathrm{d}x\leqslant\int_{\Omega_{3,M}^{+}}(\xi W)^{\frac{p}{\beta+p-1}}\cdot W^{\frac{p(\beta p-\beta-p)}{\beta+p-1}}|\nabla W|^{p}\mathrm{d}x\cdot+$$

$$\int_{\Omega_{3,M}^{-}}(\xi W)^{\frac{p}{\beta+p-1}}\cdot W^{\frac{P(\beta p-\beta-p)}{\beta+p-1}}|\nabla W|^{p}\mathrm{d}x\leqslant\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x+\int_{\Omega_{3,M}^{+}}W^{\frac{p(\beta p-\beta-p)}{\beta+p-2}}|\nabla W|^{\frac{p(\beta+p-1)}{\beta+p-2}}\mathrm{d}x$$

$$M\int_{\Omega}\xi^{p}W^{p}\mathrm{d}x+M|B_{R}|\tag{18}$$

当 M 充分大时,可使得

$$\int_{\Omega_{3,M}^{+}}W^{\frac{p(\beta p-\beta-p)}{\beta+p-2}}|\nabla W|^{\frac{p(\beta+p-1)}{\beta+p-2}}\mathrm{d}x<M|B_{R}|$$

将此式代入式(18)中得

$$\int_\Omega \xi^{p2} W^{\frac{p(\beta p-\beta-p+1)}{\beta+p-1}} |\nabla W|^p \mathrm{d}x \leqslant (M+1)\int_\Omega \xi^p W^p \mathrm{d}x + 2M|B_R| \tag{19}$$

当 M 充分大时,将式(15)(17)(19) 代入式(13) 中得

$$\int_\Omega |\nabla(\xi W)|^p \mathrm{d}x \leqslant Cr^p\left(\int_\Omega \xi^p W^p \mathrm{d}x + \int_\Omega |\nabla\xi|^p W^p \mathrm{d}x + \int_\Omega |\nabla\xi|^p \mathrm{d}x + |B_R|\right) \tag{20}$$

对式(20) 左端项应用 Sobolev 嵌入定理得

$$\left(\int_\Omega (\xi W)^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} \leqslant Cr^p\left(\int \xi^p W^p \mathrm{d}x + \int_\Omega |\nabla\xi|^p W^p \mathrm{d}x + \int_\Omega |\nabla\xi|^p \mathrm{d}x + |B_R|\right)$$

即

$$\left(\int_\Omega \xi^{p^*} \bar{u}^{\frac{Nr}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}} \leqslant Cr^p\left(\int_\Omega \xi^p \bar{u}^r \mathrm{d}x + \int_\Omega |\nabla\xi|^p \bar{u}^r \mathrm{d}x + \int_\Omega |\nabla\xi|^p \mathrm{d}x + |B_R|\right) \tag{21}$$

注意到 $\bar{u} = u+1 \geqslant 1, |B_R| \leqslant \int_{B_R} \bar{u}^r \mathrm{d}x$,由式(21) 可得

$$\left(\int_\Omega \xi^{p^*} \bar{u}^{\frac{Nr}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}} \leqslant Cr^p\left(\int_\Omega \xi^p \bar{u}^r \mathrm{d}x + + \int_\Omega |\nabla\xi|^p \bar{u}^r \mathrm{d}x + \int_{B_R} \bar{u}^r \mathrm{d}x\right) \tag{22}$$

记 $R_m = R(\theta + \frac{1-\theta}{2m})$. 取 $\xi_m \in C_0^\infty(B_{Rm}), 0 \leqslant \xi_m \leqslant 1$ 且在 B_{Rm+1} 上, $\xi_m \equiv 1$; $|\nabla\xi_m| \leqslant \frac{C}{R_m - R_{m+1}} = \frac{C\cdot 2^m}{(1-\theta)R}$. 在式(22) 中用 R_m, ξ_m 分别代替 R 及 ξ,有

$$\left(\int_{B_{Rm+1}} \bar{u}^{\frac{Nr}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}} \leqslant Cr^p\left(\int_{B_{Rm}} \bar{u}^r \mathrm{d}x + \frac{C\cdot 4^{pm}}{((1-\theta)R)^p}\int_{B_{Rm}} \bar{u}^r \mathrm{d}x\right)$$

适当放大 C 或选取 R 适当小,并由上式可得

$$\left(\int_{B_{Rm+1}} \bar{u}^{\frac{Nr}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}} \leqslant \frac{C}{((1-\theta)R)^p}\cdot 4^{pm}\cdot r^p \int_{B_{Rm}} \bar{u}^r \mathrm{d}x$$

其中 C 与 m 无关. 记 $r_m = (\frac{N}{N-p})^m \cdot p, m = 0,1,2,\cdots$. 在上式中用 r_m 代替 r,并在两边开 r_m 次方有

$$\|\bar{u}\|_{L^{r_{m+1}}(B_{Rm+1})} \leqslant \left(\frac{C}{((1-\theta)R)^p}\right)^{\frac{1}{r_m}} \cdot 4^{\frac{pm}{r_m}} \cdot r_m^{\frac{p}{r_m}} \cdot \|\bar{u}\|_{L^{r_m}(B_{Rm})}$$

迭代后得

$$\|\bar{u}\|_{L^{r_{m+1}}(B_{Rm+1})} \leqslant \left(\frac{C}{((1-\theta)R)^p}\right)^{\sum_{m=0}^{\infty}\frac{1}{r_m}} \cdot \left(4^{p\sum_{m=0}^{\infty}\frac{m}{r_m}}\right) \cdot \left(\prod_{m=0}^{\infty} r_m^{\frac{p}{r_m}}\right) \cdot \|\bar{u}\|_{L(B_R)}$$

级数 $\sum_{m=0}^{\infty}\frac{1}{r_m} = \frac{N}{p^2}, \sum_{m=0}^{\infty}\frac{m}{r_m}$ 收敛. 容易验证乘级数 $\prod_{m=0}^{\infty} r_m^{\frac{p}{r_m}}$ 收敛.

从而由上式可得

$$\|\bar{u}\|_{L^{r_m+1}(B_{Rm+1})} \leqslant \frac{C}{((1-\theta)R)\frac{N}{p}} \cdot \|\bar{u}\|_{L^p(B_{B_R})} \leqslant C\left(\fint_{B_R} \bar{u}^p \mathrm{d}x\right)$$

当 $m \to \infty$ 时, $r_{m+1} \to \infty$; $R_{m+1} \to \theta R$. 在上式中令 $m \to \infty$ 得

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \leqslant C\left(\fint_{B_R} \bar{u}^p \mathrm{d}x\right)^{\frac{1}{p}} \tag{23}$$

由 $\bar{u} = u+1$ 及 u 的非负性可得

$$\operatorname{ess\,sup}_{B_{\theta R}} u = \|u\|_{L^\infty(B_{\theta R})} \leqslant \|\bar{u}\|_{L^\infty(B_{\theta R})}$$

$$\left(\fint_{B_R} \bar{u}^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\fint_{B_R} (u+1)^p \mathrm{d}x\right)^{\frac{1}{p}} \leqslant C\left(\left(\fint_{B_R} u^p \mathrm{d}x\right)^{\frac{1}{p}} + 1\right)$$

将上述两个不等式代入式(23) 中得

$$\operatorname*{ess\,sup}_{B_{\theta R}} u \leqslant C\left(1 + \left(\fint_{B_R} u^p \mathrm{d}x\right)^{\frac{1}{p}}\right)$$

其中常数 C 仅依赖于 N、λ、λ_1、Λ_1、Λ、p、θ、diamΩ.

利用有限覆盖定理,我们有

推论 1　在定理 1 条件下,设 u 是方程(1) 的非负弱解,则对任意 $\tilde{\Omega}\subset\subset\Omega$ 有

$$\operatorname*{ess\,sup}_{\tilde{\Omega}} u \leqslant C$$

其中 C 仅赖于 N、λ、λ_1、Λ_1、Λ、p、θ、$\|\bar{u}\|_{L^p(\tilde{\Omega})}$.

推论 2　在定理 1 条件下,设 u 是方程(1) 的非负弱解,又设 $\partial\Omega\in C^2$,则有

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant C$$

其中 C 仅依赖于 N、λ、λ_1、Λ_1、Λ、p、θ、Ω、$\|u\|_{L^p(\Omega)}$.

参考文献:

[1] Nash J. Continuity of solutions of parabolic and elliptic equations[J]. Amer. J. Math. ,1958,8:931-594.

[2] 谭忠. 广义 Campan-Morrey 空间及其应用[J]. 厦门大学学报(自然科学版),1995,34(2):151-156.

[3] Gilbag D. , Trudinger N. S. Elliptic partial differential equations of second order[M]. Heidelberg, New York:Springer-Verlag,1977.

Local Extremum Principle for Weak Solutions of a Type of Elliptic Equation under Controllable Growth Conditions

HAN Pi-gong

(Dept. of Math. ,Xiamen Univ. ,Xiamen　361005,China)

Abstract: The nonlinear elliptic partial differential equation of second order, $-\mathrm{div}A(x,u,\nabla u)+B(x,u,\nabla u)=\mu$, is studied. Using Moser iterative method, the author obtains local extremum principle for weak solutions, internal and whole estimates for weak solutions, under controllable growth conditions—$A(x,z,\eta)\cdot\eta\geqslant\lambda|\eta|^p-\Lambda|z|^{p^*}-1$, $|A(x,z,\eta)|\leqslant\Lambda_1(|\eta|^{p-1}+|z|^{p^*(1-\frac{1}{p})}+1)$, $|B(x,z,\eta)|\leqslant\Lambda(|\eta|^{p(1-\frac{1}{p^*})}+|z|^{p^*-1}+1)$.

Key words: controllable growth condition; elliptic equation; weak solution