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On θ-pairs for a Class of Maximal Subgroups*

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Abstract For a maximal subgroup M of a finite group G, let $U(G:M)$ be the number of primes dividing $|G:M|$ and $U(G)$ be the largest member of all $U(G:M)$. Let $\mathscr{M}(G)$ denote the class of maximal subgroups M of G with $U(G:M)=U(G)$. By means of the θ-pairs for the maximal subgroups $M\in\mathscr{M}(G)$, we obtain in this paper some results which imply G to be solvable or supersolvable.

Keywords θ-pairs; maximal subgroup; solvable group; supersolvable group

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1 Introduction

In [1], Mukherjee and Bhattacharya introduced the concept of θ-pairs for a maximal subgroup of a finite group. The investigations on θ-pairs are continued by some authors and many results have been obtained (see [2], [3], [4], [5]). A lot of research has showed that the concept is an important tool for studying the structure of finite groups. We noticed that many results in the past required the conditions imposed on the θ-pairs for all the maximal subgroups. So it's interesting to find better conditions. In this paper, we investigate the effect on a finite group of some conditions imposed on the θ-pairs for only a class of maximal subgroups. Let $\mathscr{M}(G)=\{M \mid M$ is a maximal subgroup of G and $U(G:M)=U(G)\}$, where $U(G:M)$ denotes the number of primes dividing $|G:M|$ and $U(G)$ denotes the largest member of all $U(G:M)$. By means of the θ-pairs for the maximal subgroups $M\in\mathscr{M}(G)$, we obtain some new results which imply G to be solvable or supersolvable.

All groups considered in this paper are finite, and all unexplained notations are standard (for details, see [6]). In addition, $Core_G(M)$ denotes the core of M in G.

2 Preliminaries

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We begin with the definition of θ-pair in [1].

Definition 2.1[1] Given a maximal subgroup M of a group G, a θ-pair for M is any pair (C, D) of subgroup satisfying the following conditions:

a) $D \trianglelefteq G$, $D \leq G$;

b) $\langle M, C\rangle = G$, $\langle M, D\rangle = M$;

c) C/D has no proper normal subgroup of G/D.

Denote by $\theta(M)$ the family of all θ-pairs for M. Furthermore, we denote a partial ordering "≤" on $\theta(M)$:

$$(C, D) \leq (C', D') \text{ if and only if } C \leq C'.$$

By the definition of θ-pair, we know that $(C, D) \leq (C', D')$ also implies that $D \leq D'$. In addition, a θ-pair (C, D) is said to be maximal if there is no θ-pair (C', D') such that $C < C'$.

Lemma 2.1([3, Lemma 1]) Let G be a group. If (C, D) is a maximal θ-pair for a maximal subgroup M, put $C' = Core_G(M)\cdot C$, let $\mathscr{N}$ be a class of finite groups which is closed under taking subgroups and homomorphisms. Then the following statements hold:

(1) There exists a maximal θ-pair (C, D) such that C/D is nilpotent with Sylow 2-subgroups of class most 2.

(2) If C/D belongs to $\mathscr{N}$, then K/M_G belongs to $\mathscr{N}$ as well.

Lemma 2.2([1, Theorem 3.3]) A finite group G is solvable if and only if for every maximal subgroup M, there exists a maximal θ-pair (C, D) such that C/D is nilpotent with sylow 2-subgroups of class at most 2.

3 Main Results

Theorem 3.1 Let G be a finite group. The following statements are equivalent.

(1) G is solable.

(2) For every maximal subgroup M with $|G:M|_2 = 1$, there exists a maximal θ-pair (C, D) such that C/D is nilpotent and $C^g \nsubseteq M$ for each $g \in G$.

(3) For every maximal subgroup $M \in \mathscr{M}(G)$, there exists a maximal θ-pair (C, D) such that C/D is nilpotent with Sylow 2-subgroups of class at most 2.

Proof The equivalence of (1) and (2) is proved in [3]. We only need to sow that (3) implies (1).

If $\mho(G) = 1$, then every maximal subgroup M of G has a prime power index. By hypothesis, every maximal subgroup M of G has a maximal θ-pair (C, D) such that C/D is nilpotent with Sylow 2-subgroups of class at most 2. So G is solvable by Lemma 2.2. Hence we may let $\mho(G) > 1$ and assume that G is non-solvable. We will prove that such G does not exist.

Let $M \in \mathscr{M}(G)$. Write $N = Core_G(M)$. We first claim that the minimal normal subgroups of G/N are non-solvable.

In fact, if G/N has a minimal normal subgroup U/N which is solvable, then U/N is of prime power order, and so $|G:M| = |U/N|$ is a prime power. On the other hand, since $\mho(G) >$

1 and $\mathrm{U}(G:M)=\mathrm{U}(G)$, by the definition of $\mathscr{M}(G)$, $|G:M|$ contains at least two distinct prime divisors. This is a contradiction, which shows our claim.

Secondly, by hypothesis, there exists a maximal θ-pair (C, D) such that C/D is nilpotent with Sylow 2-subgroups of class at most 2. By Lemma 2.1, there exists a maximal θ-pair (K, M_G) such that K/M_G is nilpotent with Sylow 2-subgroups of class at most 2. Now choose a minimal normal subgroup of G/N, say L/N, such that LK has the smallest possible order. Set $E=KL$, then K is a maximal subgroup of KL. In fact, as L/N is non-solvable, then $L/N\not\leq K/N$. So $L\not\leq K$, this follows that $K<KL=E$. If K is not a maximal subgroup of KL, then there exists a maximal subgroup B of E such that $K<B<E$, this derives $K/M_G<B/M_G<KL/M_G=E/M_G$. Assume that $V/M_G\leq B/M_G$, where V/M_G is any minimal normal subgroup of G/M_G, this means $V/M_G\leq B/M_G<KL/M_G=E/M_G$. So we have $V\leq B<KL=E$ and $KV\leq KB\leq KL=E$. Then we have $KV=KL=E$. On the other hand, since $K<B$ and $V\leq B$, we have $KV\leq B<E$, a contradiction. So there is no proper minimal normal subgroup of G/M_G which contained in B/M_G, hence (B, M_G) is a θ-pair for M. But $K<B$, this contradicts with the fact that (K, M_G) is a maximal θ-pair for M. This yields that K is a maximal subgroup group of KL.

Now we have K/M_G is a maximal subgroup of $KL/M_G=E/M_G$ and K/M_G is nilpotent with Sylow 2-subgroups of class at most 2. By the famous D. J. T thoerem (see [6, IV, Satz 7.4]), $KL/M_G=E/M_G$ is solvable and hence L/M_G is solvable as well, contrary to the fact that the minimal normal subgroups of G/N are non-solvable. The proof of the theorem is now complete.

Corollary 3.2 Let G be a finite group. If for every maximal subgroup $M\in\mathscr{M}(G)$, there exists a maximal θ-pair (C, D) such that C/D is abelian, then G is solvable.

Proof Since G is abelian, it is nilpotent with sylow 2-subgroups of class at most 2. By theorem 3.1, we obtain the conclusion.

Theorem 3.3 If for every $M\in\mathscr{M}(G)$, there exists a maximal θ-pair (C, D) such that C/D is cyclic and $G=MC$, then G is either supersolvable or has a homomorhpic image isomorphic to S_4.

Proof By Theorem 3.1, G is solvable, s every maximal subgroup M of G has a prime power index, this follows that $\mathrm{U}(G)=1$. By hypothesis, every maximal subgroup of G has a maximal θ-pair (C, D) such that C/D is cyclic and $G=MC$. Suppose that G is non-supersolvable, then there exists a chief factor K/L of G which is an elementary abelian p-group of order p^n with $n>1$. Suppose that G/L is non-supersolvable. Since G/L satisfies the assumption, so we may assume that $L=1$. That is to say, K is a minimal normal subgroup of G, it is easy to prove that K is a unique minimal normal subgroups of G. Suppose that $K\leq \mathrm{H}(G)$. Since G is not supersolvable, then $G/\mathrm{H}(G)$ can not be supersolvable and by induction we can have that G has a homomorphic image isomorphic to S_4, as desired, so we can assume that $K\not\leq \mathrm{H}(G)$. So there exists a maximal subgroup M of G such that $G=MK$ and $M\cap K=1$.

If $M_G\neq 1$, as G/M_G contains a non-cyclic chief factor $KM_G/M_G\cong K$, which implies that G/M_G is non-supersolvable, by induction we may assume that $M_G=1$.

Now, by hypothesis, M has a maximal θ-pair $(C, M_G) = (C, 1)$ such that C is cyclic and $G = CM$. Write $B = M \cap C$. Then $B^G = B^{CM} = B^M \leq M$, and hence $B^G \leq M_G = 1$, namely $M \cap C = 1$. Since $G = CM$ and $M \cap C = 1$, it follows that $|C| = |G : M| = |K|$. In particular, $E = CK$ is a p-group. Moreover, it is obvious that C can not be normal in G, so we have that C is maximal in E and thus $|K : K \cap C| = |E : C| = p$. Now as K is elementary abelian and C is cyclic, it follows that K is an elementary abelian p-group of order p^2 and E is a p-group of order p^3, which contains the cyclic subgroup C of order p^2. We claim that $p = 2$. Consider the subgroup $U = M \cap E$. From $|M||E| / |M \cap E| = |G| = |M||K|$ and $|E| = p|K|$, we can see that U is cyclic of order p and $U \not\leq K$. Thus we can find two elements $u \in U$ and $v \in K$, each of which is order p such that $E = \langle u, v \rangle$. Let $[u, v] = w$. Then E has the defining relation: $u^p = v^p = w^p = 1$, $[u, v] = w$, $[u, w] = [v, w] = 1$. Therefore, if p is odd, then E is a p-group of order p^3 with exponent p, contrary to the fact that E contains the cyclic subgroup C of order p^2.

Now $p = 2$ and K is an elementary abelian 2-group of order 4 and $M_G = 1$. Consider the permutation representation of G on the four cosets of M, we can see that $G \cong S_4$. So the proof of the theorem is now complete.

Corollary 3.4　If for every maximal subgroup M of G, there exists a maximal θ-pair (C, D) such that C/D is cyclic and $G = MC$, then G is either supersolvable or has a homomorhpic image isomorphic to S_4.

References

[1] Mukherjee N P, Bhattacharya P. On theta pairs for a maximal subgroup. Proc. Amer. Math. Soc., 1990, 109: 585-596.

[2] Zhao Yaoqing. On theta pair for maximal subgroups of Finite Groups. Acta Mathematica Sinica, 1997, 40: 67-72.

[3] Li Shirong. A note on theta pairs for maximal subgroups. Comm. Algebra, 1998, 26: 4277-4284.

[4] Zhao Yaoqing. On theta pairs for maximal subgroups. comm. Algebra, 1995, 23: 2099-2106.

[5] Ballest-Boliches A, Zhao Yaoqing. On maximal subgroups of finite groups and theta pairs. Comm. Algebra, 1996, 24: 4199-4209.

[6] Huppert B. Endliche Gruppen I. Springer, Berlin, 1967.

关于一类极大子群的 θ-偶

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摘　要　对于有限群 G 的极大子群 M, 令 U$(G:M)$ 表示整除 $|G:M|$ 的素因子个数, U(G) 表示所有 U$(G:M)$ 中的最大数. 令 $\mathscr{M}(G)$ 为使得 U$(G:M)$ = U(G) 的极大子群的集合. 通过对这一类极大子群的 θ-偶赋予一定条件, 得到了判断群 G 可解、超可解的新结果.

关键词　θ-偶; 极大子群; 可解群; 超可解群