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关于次指数分布及其相关类的一个性质

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摘要:次指数及其相关类分布在随机风险理论以及概率论其它诸多领域具有广泛应用.本文利用Lebesgue-Stieltjes积分的性质,推广了Pitman的相应结果,给出了两个非负独立随机变量最小值服从指数或次指数分布的一般性充分条件.作为推论,如果两个非负独立随机变量都服从指数或次指数分布,那么其最小值也服从指数或次指数分布.

关键词:重尾分布;次指数分布;次指数性质
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1 主要结果

由于次指数及其相关类分布在随机风险理论以及概率论其它诸多领域的重要应用,近几年来它们得到了越来越广泛的重视.有关这些类型的分布性质已经被许多的学者作过深入的研究[1-10].其中两个非负独立随机变量最小值分布的次指数性在文献[1]、[8]和[10]中均有研究.需要说明的是,早在文献[8]中已有了较之文献[1]和[10]更强的结果.本文即是对文献[8]的结果作进一步的推广.

令 $\gamma\geqslant 0$,以 $S(\gamma)$ 表示具有如下性质的 $(0,+\infty)$ 上分布函数 F 的全体,

$$\lim_{x\to\infty}\frac{\overline{F^{2*}}(x)}{\overline{F}(x)}=2\int_0^\infty e^{\gamma y}\,\mathrm{d}F(y)<\infty \tag{1}$$

且

$$\lim_{x\to\infty}\overline{F}(x-y)/\overline{F}(x)=e^{\gamma y},\ \forall y\in R \tag{2}$$

这里 F^{2*} 表示 F 的二重卷积,$\overline{F}(x)=1-F(x)$.

特别,$S:=S(0)$ 称为次指数分布类,此为重要的重尾分布子类.当 $\gamma>0$ 时,$S(\gamma)$ 为重要的轻尾分布子类.以 $\mathscr{L}(\gamma)$ 表示满足条件(2)的分布函数类.

设 X,Y 为两个非负的随机变量,且相互独立.记 X,Y 和 $\min\{X,Y\}$ 的分布分别为 $F_X,F_Y,F_{\min}$.在文献[8]中,Pitman证明了如下结论:

命题1 如果 $F_X\in S,F_Y\in S$,则 $F_{\min}\in S$.

但是,文献[10]和[1]却在额外附加的条件下证明了上述命题,这些条件是不必要的.本文改进了命题1的结果,一方面将Pitman[8]的结论推广到分布类 $S(\gamma)$ 的情形,另一方面减弱了结论的条件.本文所得到的主要定理如下:

定理1 设 $F_X\in S(\gamma_1),F_Y\in\mathscr{L}(\gamma_2),\gamma_1,\gamma_2\geqslant 0$,而且

$$\limsup_{x\to\infty}\frac{\overline{F_Y^{2*}}(x)}{\overline{F_Y}(x)}<\infty \tag{3}$$

则

$$F_{\min}\in S(\gamma_1+\gamma_2) \tag{4}$$

作为定理1的推论,易知:

推论1 设 $F_X\in S(\gamma_1),F_Y\in S(\gamma_2),\gamma_1,\gamma_2\geqslant 0$,则

$$F_{\min}\in S(\gamma_1+\gamma_2).$$

易知命题1是上述推论的一个特例.

2 定理1的证明

引理1 设 $\gamma\geqslant 0$ 且 $\int_0^\infty e^{\gamma y}\,\mathrm{d}F(y)<\infty$,则 $F\in S(\gamma)$ 等价于 $F\in\mathscr{L}(\gamma)$,且

$$\lim_{A\to\infty}\lim_{x\to\infty}\frac{\int_A^{x-A}\overline{F}(x-y)\,\mathrm{d}F(y)}{\overline{F}(x)}=0 \tag{5}$$

证明 此引理的证明可由文献[7]之定理2.1的证明得到.

根据文献[7],在下文所涉及的类 $\mathscr{L}(\gamma)$ 中分布性质的证明时,我们均可假定其为连续分布而不失结论的一般性,这就保证了下文所用的Lebesgue-Stieltjes积分分部积分公式的形式正确性.

定理1的证明 显然

$$\overline{F_{\min}}(x)=\overline{F_X}(x)\,\overline{F_Y}(x),\ \forall x\in R \tag{6}$$

于是由 $F_X\in\mathscr{L}(\gamma_1)$ 及 $F_Y\in\mathscr{L}(\gamma_2)$,可知

$$F_{\min}\in\mathscr{L}(\gamma_1+\gamma_2) \tag{7}$$

又

$$\overline{F_Y^{2*}}(x)=\overline{F_Y}(x)+\int_0^x\overline{F_Y}(x-y)\,\mathrm{d}F_Y(y) \tag{8}$$

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于是由式(3) 及 Fatou 引理, 有

$$\int_0^\infty e^{\gamma_2 y}\mathrm{d}F_Y(y) \leqslant \liminf_{x\to\infty}\frac{\overline{F_Y^{2*}}(x)}{\overline{F_Y}(x)} - 1 \leqslant \limsup_{x\to\infty}\frac{\overline{F_Y^{2*}}(x)}{\overline{F_Y}(x)} - 1 < \infty \tag{9}$$

于是由式(6), 有

$$\int_0^\infty e^{(\gamma_1+\gamma_2)x}\mathrm{d}F_{\min}(x) = -\int_0^\infty e^{(\gamma_1+\gamma_2)x}\mathrm{d}\,\overline{F_{\min}}(x) =$$

$$-\int_0^\infty e^{(\gamma_1+\gamma_2)x}\mathrm{d}[\overline{F_X}(x)\,\overline{F_Y}(x)] =$$

$$-\int_0^\infty e^{(\gamma_1+\gamma_2)x}\,\overline{F_Y}(x)\,\mathrm{d}\,\overline{F_X}(x) - \int_0^\infty e^{(\gamma_1+\gamma_2)x}\,\overline{F_X}(x)\,\mathrm{d}\,\overline{F_Y}(x) =$$

$$\int_0^\infty e^{(\gamma_1+\gamma_2)x}\,\overline{F_Y}(x)\,\mathrm{d}F_X(x) + \int_0^\infty e^{(\gamma_1+\gamma_2)x}\,\overline{F_X}(x)\,\mathrm{d}F_Y(x) \leqslant$$

$$\sup_{x>0}\left\{e^{\gamma_2 x}\,\overline{F_Y}(x)\right\}\int_0^\infty e^{\gamma_1 x}\mathrm{d}F_X(x) + \sup_{x>0}\left\{e^{\gamma_1 x}\,\overline{F_X}(x)\right\}\int_0^\infty e^{\gamma_2 x}\mathrm{d}F_Y(x) < \infty \tag{10}$$

由式(7)、(10) 和引理 1 知, 只要证明下述结论, 即可完成定理的证明,

$$\lim_{A\to\infty}\lim_{x\to\infty}\frac{\int_A^{x-A}\overline{F_{\min}}(x-y)\,\mathrm{d}F_{\min}(y)}{\overline{F_{\min}}(x)} = 0 \tag{11}$$

由式(6) 可得

$$\frac{\int_A^{x-A}\overline{F_{\min}}(x-y)\,\mathrm{d}F_{\min}(y)}{\overline{F_{\min}}(x)} = -\frac{\int_A^{x-A}\overline{F_{\min}}(x-y)\,\mathrm{d}\,\overline{F_{\min}}(y)}{\overline{F_{\min}}(x)} =$$

$$-\frac{\int_A^{x-A}\overline{F_X}(x-y)\,\overline{F_Y}(x-y)\,\mathrm{d}[\overline{F_X}(y)\,\overline{F_Y}(y)]}{\overline{F_X}(x)\,\overline{F_Y}(x)} =$$

$$-\frac{\int_A^{x-A}\overline{F_X}(x-y)\,\overline{F_Y}(x-y)\,\overline{F_Y}(y)\,\mathrm{d}\,\overline{F_X}(y)}{\overline{F_X}(x)\,\overline{F_Y}(x)} - \frac{\int_A^{x-A}\overline{F_X}(x-y)\,\overline{F_Y}(x-y)\,\overline{F_X}(y)\,\mathrm{d}\,\overline{F_Y}(y)}{\overline{F_X}(x)\,\overline{F_Y}(x)} =$$

$$\frac{\int_A^{x-A}\overline{F_X}(x-y)\,\overline{F_Y}(x-y)\,\overline{F_Y}(y)\,\mathrm{d}F_X(y)}{\overline{F_X}(x)\,\overline{F_Y}(x)} + \frac{\int_A^{x-A}\overline{F_X}(x-y)\,\overline{F_Y}(x-y)\,\overline{F_X}(y)\,\mathrm{d}F_Y(y)}{\overline{F_X}(x)\,\overline{F_Y}(x)} :=$$

$$\text{iv} + ㊆ \tag{12}$$

记

$$G_X(t) = \frac{\int_A^t \overline{F_X}(x-y)\,\mathrm{d}F_X(y)}{\overline{F_X}(x)} \tag{13}$$

且

$$G_Y(t) = \frac{\int_A^t \overline{F_Y}(x-y)\,\mathrm{d}F_Y(y)}{\overline{F_Y}(x)} \tag{14}$$

由引理 1 知

$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A) = 0 \tag{15}$$

由式(8) 和(3) 知

$$\limsup_{A\to\infty}\limsup_{x\to\infty}G_Y(x-A) < \infty \tag{16}$$

令 $x > 2A$, 由分部积分公式, 得

$$\text{iv} = \frac{\int_A^{x-A}\overline{F_Y}(x-y)\,\overline{F_Y}(y)\,\mathrm{d}G_X(y)}{\overline{F_Y}(x)} = \frac{\overline{F_Y}(A)\,\overline{F_Y}(x-A)G_X(x-A) - \int_A^{x-A}G_X(y)\,\mathrm{d}[\overline{F_Y}(x-y)\,\overline{F_Y}(y)]}{\overline{F_Y}(x)} =$$

$$\frac{\overline{F_Y}(A)\,\overline{F_Y}(x-A)G_X(x-A) + \int_A^{x-A}[G_X(y) - G_X(x-y)]\,\overline{F_Y}(x-y)\,\mathrm{d}\,\overline{F_Y}(y)}{\overline{F_Y}(x)} =$$

$$\frac{\overline{F_Y}(A)\,\overline{F_Y}(x-A)G_X(x-A)}{\overline{F_Y}(x)} + \int_A^{x-A}[G_X(y) - G_X(x-y)]\,\mathrm{d}\,\overline{G_Y}(y) \leqslant$$

$$\frac{\overline{F_Y}(A)\,\overline{F_Y}(x-A)G_X(x-A)}{\overline{F_Y}(x)} + G_X(x-A)G_Y(x-A) \tag{17}$$

上式中的最后一个不等式成立是由于

$$G_X(y) - G_X(x-y) \leqslant G_X(x-A),\ \forall y \in (A, x-A) \tag{18}$$

同理可得

$$㊆ \leqslant \frac{\overline{F_X}(A)\,\overline{F_X}(x-A)G_Y(x-A)}{\overline{F_X}(x)} + G_X(x-A)G_Y(x-A) \tag{19}$$

由式(9) 及 $\int_0^{\infty} e^{\gamma_1 x}\,\mathrm{d}F_X(x) < \infty$, 可得

$$\lim_{A\to\infty}\left(e^{\gamma_2 A}\,\overline{F_Y}(A)\right) = \lim_{A\to\infty}\left(e^{\gamma_1 A}\,\overline{F_X}(A)\right) = 0 \qquad (20)$$

于是由上式及式(15) ~ (20) 推得

$$0 \leqslant \limsup_{A\to\infty}\limsup_{x\to\infty} \text{iv} \leqslant$$
$$\lim_{A\to\infty}\left(\overline{F_Y}(A)\lim_{x\to\infty}\frac{\overline{F_Y}(x-A)}{\overline{F_Y}(x)}\right)\times$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)+$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)\limsup_{A\to\infty}\limsup_{x\to\infty}G_Y(x-A) =$$
$$\lim_{A\to\infty}\left(e^{\gamma_2 A}\,\overline{F_Y}(A)\right)\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)+$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)\times$$
$$\limsup_{A\to\infty}\limsup_{x\to\infty}G_Y(x-A) = 0 \qquad (21)$$

以及

$$0 \leqslant \limsup_{A\to\infty}\limsup_{x\to\infty} ㊆ \leqslant$$
$$\lim_{A\to\infty}\left(\overline{F_X}(A)\lim_{x\to\infty}\frac{\overline{F_X}(x-A)}{\overline{F_X}(x)}\right)\times$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_Y(x-A)+$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)\times$$
$$\limsup_{A\to\infty}\limsup_{x\to\infty}G_Y(x-A) =$$
$$\lim_{A\to\infty}\left(e^{\gamma_1 A}\,\overline{F_X}(A)\right)\lim_{A\to\infty}\lim_{x\to\infty}G_Y(x-A)+$$
$$\lim_{A\to\infty}\lim_{x\to\infty}G_X(x-A)\times$$
$$\limsup_{A\to\infty}\limsup_{x\to\infty}G_Y(x-A) = 0 \qquad (22)$$

由式(12)、(21) 及(22) 即可知式(11) 成立, 证毕.

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On a Property of Subexponential Distributions and Related Classes

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Abstract: Subexponential distribution and the correlative distribution is applied extensively in the theory of probability and stochastic risk. In this paper, the Lebesgue-Stieltjes integral was used to generalize the result in Pitmam. Let X and Y be independent nonnegative random variables. A sufficient condition was given under which the distribution of the random variable $\min\{X, Y\}$ belongs to the class of exponential or subexponential distributions. The sufficient condition is satisfied if the distribution of the random variable X as well as Y belongs to the class of exponential or subexponential distributions.

Key words: heavy-tailed distribution; subexponential distribution; subexponential property