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# Restarted Simpler GMRES Augmented with Approximate Errors

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**Abstract** This paper describes a simpler form of restarted LGMRES, augmented with several previous approximate errors. This is based on the fact that the residual vectors at the end of each restart cycle of restarted simpler GMRES also alternate direction in a cyclic fashion, similar to the case for restarted GMRES. The advantage for the new variant is that it requires less amount of work than restarted LGMRES and the experiments results show that it has better performance.

**Key words** Simpler GMRES; restart; Krylov subspace methods; LGMRES
**CLC number** O 241 **Document code** A

## 1 Introduction

An important class of iterative methods for solving the large sparse linear system

$$Ax = b. \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix and $x$, $b$ are vectors of length $n$, is Krylov subspace methods. At present, there exists a large variety of Krylov subspace methods, such as GMRES [5], BiCG and QMR [2]. GMRES is the most popular choice which finds the aproximate solution in the Krylov subspace that minimizes the 2-norm of the residual.

GMRES has the drawback that the orthogonalizaion costs per iteration grows dramatically with the increasing of the dimension of the subspace. Then a well-known variant suggested [5] is to restart the algorithm once the Krylov subspace reaches dimension $m$. However, restarting generally slows the convergence due to the loss of orthogonality to previously generated subspace at each restart.

Morgan proposed new methods to retain the information at each restart by augmenting the Krylov space by approximate eigenvectors [3]. These methods have been shown to improve the convergence of GMRES for problems with small eigenvalues. In [1], a new technique of augmenting the Krylov subsapce with approximate errors was proposed. The method was referred to

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as LGMRES. It can accelerate the convergence of GMRES for a wide range of problems. As we know simpler GMRES described by Walker and Zhou [6] is cheaper to implement than GMRES. In this paper we presented simpler form of LGMRES and we can see the improvement of performance.

An outline of this paper is as follows: In Section 2 we briefly describe the GMRES and simpler GMRES(SGMRES) algorithm. In Section 3 we describe the LGMRES method and the simpler implementation. In Section 4 we describe numerical experiments for a large amount of problems and draw comparisons among different variants for GMRES.

# 2 GMRES and SGMRES

## 2.1 The GMRES method

GMRES finds the approximate solution in the Krylov subspace that minimizes the 2-norm of the residual. Let $x_0$ be an initial guess to (1) and let $r_0 = b - Ax_0$ be the initial residual vector. GMRES uses Arnoldi's algorithm to construct an orthonormal basis $V_m = [v_1, v_2, \cdots, v_m]$ for Krylov subspace $K_m(A, r_0) = span\{r_0, Ar_0, \cdots, A^{m-1}r_0\}$ and this results in the factorization $AV_m = V_{m+1}\bar{H}_m$ where $\bar{H}_m \in R^{(m+1)\times m}$ is upper-Hessenberg. The generic restarted GMRES is described as follows.

The following is a version of the restarted GMRES algorithm.

**Algorithm 1** The restarted $m$-step GMRES algorithm

1. Start: Choose a stopping criterion $\epsilon$ and an initial guess $x_0$, compute $r_0 = b - Ax_0$, $\beta = \|r_0\|_2$ and $v_1 = r_0/\|r_0\|_2$. Set the size of the Krylov subspace to be $m$.

2. Iteration:

for $j = 1: m$ do: $v = Av_j$;

for $i = 1: j$ do: $h_{i,j} = (v, v_i)$;

$v = v - h_{i,j}v_i$;

end

$h_{j+1,j} = \|v\|_2$; $v_{j+1} = v/h_{j+1,j}$;

end

3. Form the approximate solution: Compute $y_m$ which minimizes $\|\beta e_1 - \bar{H}_m y\|_2$ and $x_m = x_0 + V_m y_m$.

4. Restart: Compute the residual vector $r_m = b - Ax_m$, and relative residual norm $\frac{\|r_m\|_2}{\|b\|_2}$, if satisfied then stop, else let $x_0 = x_m$, and go to 1.

The details on the practical implementation of the algorithm above are discussed in [5].

## 2.2 Simpler GMRES

When the Arnoldi process is applied with $v_1 = Ar_0/\|Ar_0\|_2$, the simpler GMRES implementation (SGMRES) results. In this case, the Arnoldi process produces an orthonormal basis $\{v_1, v_2, \cdots, v_m\}$ for the Krylov subspace $K_m(A, v_1) = span\{v_1, Av_1, \cdots, A^{m-1}v_1\}$. At the end of each iteration, if $x_{m-1}$ is not the solution of (1), then $K_m(A, v_1) = A(K_m(A, r_0))$ has dimension $m$. [6]

The SGMRES algorithm is described as follows.

**Algorithm 2**　Simpler GMRES

1. Initialization: Given $x_0$ and the stopping criteria $\epsilon$. Computer $r = b - Ax_0$ and $\rho_0 = \| r \|_2$. If $\rho_0 < \epsilon$, accept $x_0$ and exit, else set $r = r/\rho_0$, $\rho = 1$.

2. Iterate: for $k = 1: m$ do:

(i) Evaluate $v_k = Av_{k-1}(v_1 = Ar)$.

(ii) If $k > 1$ then set $\rho_{ik} = v_i^T v_k$ and update $v_k = v_k - \rho_{ik} v_i$ for $i = 1: k-1$.

(iii) set $\rho_{kk} = \| v_k \|_2$ and $v_k = v_k / \rho_{kk}$.

(iv) set $R_k = \begin{pmatrix} R_{k-1} & & & \rho_{1k} \\ & & & \vdots \\ 0 & \cdots & 0 & \rho_{kk} \end{pmatrix}$ $(R_1 = (\rho_{11}))$

(v) set $\xi_k = r^T v_k$, update $\rho = \rho \sin(\cos^{-1}(\xi_k/\rho))$. If $\rho \cdot \rho_0 \le \epsilon$, go to step 3.

(vi) update $r = r - \xi_k v_k$.

3. Solve: Let $k$ be the final iteration number from Iterate.

(i) Solve $R_k y = (\xi_1, \cdots, \xi_k)^T$ for $y = (\eta_1, \cdots, \eta_k)^T$

(ii) form $z = \begin{cases} \eta_1 r & (\text{if } k = 1) \\ \eta_1 r + \sum_{i=1}^{k-1} (\eta_{i+1} + \eta_1 \xi_i) v_i & (\text{if } k > 1) \end{cases}$

(iii) update $x = x + \rho_0 Z$.

(iv) if $\rho \cdot \rho_0 \le \epsilon$, accept $x$ and exit, else update $r = (r - \xi_k v_k)/\rho$, $\rho_0 = \rho \cdot \rho_0$, $\rho = 1$ and go to step 2.

Details on the implementation of the above algorithm is discussed in [6].

This algorithm results in the following decomposition $A[r_0\ W_{m-1}] = W_m R_m$, where $R_m$ is an $m \times m$ upper triangular matrix. Then the problem reduces to an upper triangular least-square problem instead of an upper Hessenberg least-square problem and it is generally cheaper to implement.

# 3　LGMRES and Simpler LGMRES

In [1], a new technique, referred to as LGMRES, was presented to accelerate the convergence of restarted GMRES. The authors observed that the residual vetors at the end of each restart cycle of restarted GMRES often alternate direction in a cyclic fashion, thereby slowing convergence. For simpler GMRES, there exists similar phenomenon. In this paper we propose a simpler form of LGMRES.

## 3.1　Motivation

Consider restarted simpler GMRES when solving problem (1). We refer to the group of $m$ iterations between successive restarts as a cycle. The restarted number is denoted with a subscript: $r_i$ is the residual after $i$ cycles or $m \times i$ iterations. The residual at the end of cycle $i+1$ is a polynomial in $A$ times the residual from the previous cycle, $r_{i+1} = p_{i+1}^{m}(A) r_i$, where $p_{i+1}^{m}(A)$ is the degree $m$ residual polynomial. During each restart cycle $i$, SGMRES($m$) finds $x_{i+1} \in x_i + K_m(A, r_i)$ such that $r_{i+1} \perp AK_m(A, r_i) = K_m(A, v_1)$ $(v_1 = Ar_i / \| Ar_i \|_2)$.

As mentioned in the introduction, GMRES($m$) (SGMRES($m$)) does not maintain orthogonality between approximation spaces generated at successive restarts. As a result, slow convergence or even stagnation can occur. In the case of slow convergence, like in GMRES($m$), we have also observed in SGMRES($m$) that the residual vectors point in nearly the same direction at the end of every other restart cycle. In other words, the angle between $r_{i+1}$ and $r_{i-1}$ is samll and $r_{i+1} \approx \alpha r_{i-1}$. The angles between every other residual vector are referred to as skip angles, e. g. $\angle(r_{i+1}, r_{i-1})$, and the angles between consecutive restart cycles as sequential angles.

For many problems, it is found that skip angles are relatively small even when the sequential angles are a reasonable size (i. e, stagnation is not occuring). We also test results for SGMRES (30) on the four examples available from the Matrix Market Collection. The number of iteration required for convergence ( $\| r_i \|_2 / \| r_0 \|_2 \leq 10^{-9}$) as well as the median sequential and median skip angle values are listed in Table 1. The low skip angle values appear to indicate that faster convergence should be possible if some degree of orthogonality to previous approximation spaces were maintained.

Table 1 Results for SGMRES(30). Problem size, outer iterations required for $\| r_i \|_2 / \| r_0 \|_2 \leq 10^{-9}$, median skip angle, and median sequential angle are listed for each problem.

| Problem | Size(n) | Iterations | $\angle(r_i, r_{i-1})$ | $\angle(r_{i+1}, r_{i-1})$ |
|---|---|---|---|---|
| add20 | 2395 | 28 | 49.97 | 1.16 |
| orsirr1 | 1030 | 188 | 19.45 | 4.79 |
| orsreg1 | 2205 | 26 | 60.95 | 12.97 |
| sherman1 | 1000 | 95 | 27.32 | 0.12 |

The motivation for LGMRES came from a desire to prevent the alternating behavior observed for GMRES($m$) which results in repetitive information in successive restart cycle. Suppose that $\hat{x}$ is the true solution to problem (1). The error after the ith restart cycle of GMRES($m$) is denoted by $e_i$, where $e_i = \hat{x} - x_i$.

If the approximation space contains the exact correction $e_i$ such that $\hat{x} = x_i + e_i$, then we have solved the problem. Define $z_i = x_i - x_{i-1}$ as the approximation to the error after the ith GMRES ($m$) restart cycle, and $z_j \equiv 0$ for $j < 1$. This error approximation vector serves as the choice of vector with which to augment the next approximation space $K_m(A, r_i)$. Note that $z_i \in K_m(A, r_{i-1})$. Therefor, this error approximation $z_i$ in some sense represents the space $K_m(A, r_{i-1})$ generated in the previous cycle and subsequently discarded and is a natural choice of vector with which to augment the next approximation space $K_m(A, r_i)$.

LGMRES($m$, $k$) augments the standard Krylov approximation space with $k$ previous approximations to the error. Therefore, at the end of restart cycle $i+1$, LGMRES($m$, $k$) finds an approximate solution to (1) in the following way:

$$x_{i+1} = x_i + q_{i+1}^{m-1}(A) r_i + \sum_{j=i-k+1}^{i} \alpha_j z_j$$

where polynomial $q_{i+1}^{m-1}$ and $\alpha_j$ are chosen such that $\| r_{i+1} \|_2$ is minimized. Note that $k=0$ corresponds to standard GMRES($m$).

As we know, simpler GMRES($m$) is cheaper to implement than GMRES($m$) and as mentioned previously there exists similar alternating phenomenon for simpler GMRES($m$), we present a simpler implementation for LGMRES. To describe the simpler LGMRES algorithm, first we present the augmented simpler Arnoldi process.

## 3.2 Algorithm and properties

Suppose we want to augment the Krylov subspace $K_m(A, r_i)$ with $k$ previous approximate errors $z_i, z_{i-1}, \cdots, z_{i-k+1}$ at the cycle $i+1$. Define $Z_k = [z_i, z_{i-1}, \cdots, z_{i-1+1}]$. For convenience we denote the augmented simpler Arnoldi process by S-Arnoldi($m-k$, $k$). The S-Arnoldi($m-k$, $k$) algorithm is described as follows.

**Algorithm 3 Augmented Simpler Arnoldi (S-Arnoldi($m-k, k$))**

1. Start: Input $k$ approximate errors $Z_k$, $\tilde{Z}_k = AZ_k$, $m$ and $k$. Compute $v_1 = Ar_i$, $R_{11} = \| v_1 \|_2$, $v_1 = v_1 / R_{11}$.

2. Iterate: For $j = 2:m$

(i) if $j \le m-k$; $v_j = Av_{j-1}$; *else* $v_j = \tilde{z}_{i-(j-m-1)}$.

(ii) for $k = 1:j-1$

$R_{kj} = v_j^T v_k$, $v_j = v_j - R_{kj} v_k$.

$R_{jj} = \| v_j \|_2$, $v_j = v_j / R_{jj}$.

3. output: If $k=0$, $Y_m = [r_i \; V_{m-1}]$, *else* $Y_m = [r_i \; V_{m-k-1} \; Z_k]$.

The result of the above algorithm is the decomposition

$$AY_m = V_m R_m \tag{2}$$

where $R_m$ is an $m \times m$ upper triangular matrix.

Let $\prod_m$ denotes orthogonal projection onto $AK_m(A, r_i) = K_m(A, v_1)$. Since $V_m = [v_1, \cdots, v_m]$ is an orthonormal basis of $K_m(A, v_1)$, $\prod_m = V_m V_m^T$. Denote by $r_m^{(i+1)}$ the residual at the cycle $i+1$ when the approximation subspace is dimension of $m$. Note that $r_m^{(i+1)} = r_{i+1}$. Then $r_m^{(i+1)} = (I - \prod_m) r_i$. [4]

So

$$r_m^{(i+1)} = r_i - [v_1 \, v_2 \cdots v_m] \begin{pmatrix} v_1^T r_i \\ v_2^T r_i \\ \vdots \\ v_m^T r_i \end{pmatrix}$$

Then

$$r_i = r_k^{(i+1)} + [v_1 v_2 \cdots v_m]\omega_n = r_k^{(i+1)} + \sum_{j=1}^{m}\xi_j v_j$$

where $\omega_n = \begin{pmatrix} v_1^T r_i \\ v_2^T r_i \\ \vdots \\ v_m^T r_i \end{pmatrix} = (\xi_1, \xi_2, \cdots, \xi_m)^T$. Then we get the recurrence $r_m^{(i+1)} = r_{m-1}^{(i+1)} - \xi_m v_m$. The residual norm can be easily updated using

$$\| r_m^{(i+1)} \|_2 = \sqrt{\| r_{m-1}^{(i+1)} \|_2 - \xi_m^2} = \| r_{m-1}^{(i+1)} \|_2 \sin(\cos^{-1}(\xi_m / \| r_{m-1}^{(i+1)} \|_2)) \quad (3)$$

According to the minimal residual criterion, we have

$$b - Ax_m \perp AK_m(A, r_i) = K_m(A, v_1) \quad (4)$$

Let $x_m = x_0 + \delta_m$, where $\delta_n$ is the correction. Then $V_m^T(b - A(x_0 + \delta_m)) = 0$, or further $\omega_n = V_m^T r_i = V_m^T A\delta_m = V_m^T A Y_m \hat{y} = R_m \hat{y}$. Let $\hat{y} = R_m^{-1}\omega_n = (\eta_1, \eta_2, \cdots, \eta_m)^T$. Then the correction is given by $\delta_m = Y_m \hat{y}$ where

$$\delta = \begin{cases} r_i \hat{y}_1 & \text{if } l = 1 \\ (r_i, v_1, \cdots, v_{l-1})\hat{y}_l & \text{if } 1 < l \le m - k \\ (r_i, v_1, \cdots, v_{m-k-1}, z_i, \cdots, z_{i+m-k-l+1})\hat{y}_l & \text{if } m - k < l \le m \end{cases} \quad (5)$$

According to the recurrence $r_m^{(i+1)} = r_{m-1}^{(i+1)} - \xi_m v_m$, the correction can be written as

$$\delta_l = \begin{cases} \eta_1 r_i \hat{y}_1 & \text{if } l = 1 \\ \eta_1 r_{l-1}^{(i+1)} + \sum_{j=1}^{l-1}(\eta_{j+1} + \eta_1\xi_j)v_j & \text{if } 1 < l \le m - k \\ \eta_1 r_{l-1}^{(i+1)} + \sum_{j=1}^{m-k-1}(\eta_{j+1} + \eta_1\xi_j)v_j + \sum_{j=m-k}^{l-1}(\eta_1\xi_j v_j + \eta_{j+1} z_{i-(j-m+k)}) & \\ & \text{if } m - k < l \le m \end{cases} \quad (6)$$

Based on the implementation of augmented simpler Arnoldi process, we can describe the simpler LGMRES algorithm now. Suppose we have got $k$ approximate errors $Z_k = [z_i, z_{i-1}, \cdots, z_{i-k+1}]$ and the residual $r_i$ at the end of the cycle $i$. The restart cycle $i+1$ is described as follows.

**Algorithm 4** Simpler LGMRES for restart cycle $i+1$

1. Apply augmented simpler Arnoldi with $Z_k$, $r_i$ and $A$ to generate $Y_m$, $V_m$, $R_m$. Find $\tilde{\omega}_n = [\xi_1, \xi_2, \cdots, \xi_m]^T$ and $r_m^{(i+1)} = r_i - V_m\tilde{\omega}_n$ during the simpler Arnoldi orthogonalization, that is, at each jth orthogonalization step we find $\xi_j = v_j^T r_{j-1}$ and $r_j = r_{j-1} - \xi_j\omega$, we also update $\rho$ using $\rho = \rho\sin(\cos^{-1}(\xi_j)/\rho)$ and if $\rho\rho_0 \le \epsilon$, go to the next step.

2. Form the approximate solution. $\hat{y} = R_m^{-1}\tilde{\omega}_n$ and $x_m = x_0 + \rho_0 Y_m\hat{y} = x_0 + \rho_0\delta_m$ where $\delta_m$ is given by (6).

3. Judge whether restarting is necessary. If $\rho\rho_0 \le \epsilon$, accept $x_m$ and exit, else $x_0 = x_m$, $\rho_0 = \rho\rho_0$ and $r_{i+1} = r_m^{(i+1)}/\rho_0$. Update $Z_k = [z_{i+1}, z_i, \cdots, z_{i-k+2}]$ where $z_{i+1} = \delta_m$. Set $\rho = 1$, $i = i + 1$ and go to step 1.

Note that only $i-1$ error approximations are available at the beginning of restart cycles with $i \le k$ because $z_j = 0$ when $j < 1$. Therefore, additional Arnoldi vectors are used to replace $z_j$ when

$j<1$ so that the approximation space is of dimension $m$ for each cycle. In other words, the first cycle $i=1$ of simpler LGMRES($m-k$, $k$) is equivalent to the first cycle of SGMRES($m$).

As mentioned previously, (simpler) LGMRES was designed to prevent the alternating behavior for the residual vectors. Now we compare the skip and sequential angles for SGMRES($m$) and simpler LGMRES($m$, $k$). There exists the same conclusion as in [3] for GMRES($m$) and LGMRES($m$, $k$) and the proofs are similar. We present the main results in terms of simpler LGMRES.

**Theorem** 1[1] (sequential angles) Let $r_{i+1}$ and $r_i$ be the residuals from SGMRES restart cycle $i+1$ and $i$, respectively. Then the angle between these residuals is given by

$$\cos\angle(r_{i+1}, r_i) = \frac{\| r_{i+1} \|_2}{\| r_i \|_2} \tag{7}$$

Note that the results holds for GMRES($m$) and (simpler) LGMRES($m$, $k$). The above theorem indicates that the convergence rate correlates to the size of the angles between consecutive residual vectors. If consecutive residual vectors are nearly orthogonal to each other, then convergence is fast. If we find an $r_{i+1}$ such that $r_{i+1}\perp r_i$, then we have found the exact solution.

The following result holds for both GMRES($m$) and SGMRES($m$).

**Theorem** 2[1] (skip angles) Let $r_{i+1}$ and $r_{i-1}$ be the residuals from SGMRES restart cycles $i+1$ and $i-1$, respectively. Then the angle between these residuals is given by

$$\cos\angle(r_{i+1}, r_{i-1}) = \frac{\| r_{i+1} \|_2}{\| r_{i-1} \|_2} - \frac{(A\delta_{i+1}, A\delta_i)}{\| r_{i+1} \|_2 \| r_{i-1} \|_2} \tag{8}$$

where $r_{i+1}= r_i- A\delta_{i+1}$ and $r_i= r_{i-1}- A\delta_i$.

In terms of describing convergence, the above result is not immediately helpful. Next we can get a few of its implications after giving a corresponding result for simpler LGMRES.

**Theorm** 3 (every other residual vector) Let $r_{i+1}$ and $r_{i-1}$ be the residuals from simpler LGMRES restart cycles $i+1$ and $i-1$, respectively. Then the angle between these residuals is given by

$$\cos\angle(r_{i+1}, r_{i-1}) = \frac{\| r_{i+1} \|_2}{\| r_{i-1} \|_2} \tag{9}$$

**Proof** Similar to the proof of Theorem 6 in [1]. The only difference is the current Krylov approximation space $\Omega= K_m(A, r_i) \cup span\{z_j\}_{j=(i-k+1):i}$, where $K_m(A, r_i)= span\{r_i, v_1, v_2, \cdots, v_{m-1}\}$ instead of $K_m(A, r_i)= span\{v_1, v_2, \cdots, v_m\}$ in the case of Theorem 6 in [1].

This results indicates that, for simpler LGMRES, the progress of the iteration also correlates with the skip angles. Therefore, fast convergence implies large skip angles. When a problem exhibits signs of alternating residuals with SGMRES($m$), then the angle between $r_{i-1}$ and $r_{i+1}$ is small. In this case, since $A\delta_{i+1}= r_i- r_{i+1}$ and $A\delta_i= r_{i-1}- r_i$, then the term $(A\delta_{i+1}, A\delta_i)$ in Theorem 2 is negative. We have observed this result in our experiments. Since simpler LGMRES appends a previous error approximation to the approximation space during cycle $i+1$, the term $(A\delta_i, A\delta_{i-1})$ became zero by construction. In the next section we show that this simpler LGM-

RES augmenting scheme tends to increase the skip angle over that of SGMRES($m$) and prevents the alternating behavior often observed in restarted SGMRES.

## 4 Numerical Experiments

In this section we describe the results of numerical experiments using Matlab 7.0. All the numerical tests have been carried out on an Intel(R) Celeron(R) 2.66GHz with main memory 256MB and the machine precision $eps = 2.22\times 10^{-16}$ on a Window XP-based system. We compare restarted simpler LGMRES($m-k$, $k$) with SGMRES($m$) to see the acceleration of convergence. We also compare simpler LGMRES with LGMRES to see the improvement in performance.

We look at a test set of 10 problems from the Matrix Market Collection. These problems include the following: add20, orsirr1, orsreg1, sherman1, sherman4, cavity05, cavity10, cdde1, e05r0000 and nos3. All the right-hand sides are constructed by setting the exact solution to be the vector whose elements are all ones. The initial guess is chosen to be zero vector. For all the exmples, the dimension $m$ of the approximation subspace is set to be 30. In all the tables, size ($n$) denotes the order of the test matrix, iter the number of restarts, mv the number of matrix-vector multiplications and time(sec.) the CPU running time in seconds for sloving the problem. The stopping criteria is set to be le-9. The maximum number of outer iterations is set to be 300 for all the examples and the symbol-mean that the algorithm does not converge within 300 outer iterations.

Experimentally, we observe that simpler LGMRES($m-k,k$) nearly always has a larger median skip angle than does SGMRES($m$). For example, in Table 2 we list the LGMRES(29, 1) results for the same four problems for which SGMRES(30) results were provided in Talbe 1 in section 3.1.

Table 2 Results for simpler LGMRES(29, 1). Problem size, outer iterations required for $\|r_i\|_2/\|r_0\|_2\le 10^{-9}$, median skip angle, and median sequential angle are listed for each problem.

| Problem | Size(n) | Iterations | $\angle(r_i, r_{i-1})$ | $\angle(r_{i+1}, r_{i-1})$ |
|---|---|---|---|---|
| add20 | 2395 | 16 | 59.65 | 89.58 |
| orsirr1 | 1030 | 71 | 43.27 | 83.35 |
| orsreg1 | 2205 | 14 | 68.46 | 76.16 |
| sherman1 | 1000 | 23 | 63.10 | 88.37 |

For most of the test problems, $k=1$ is the best for the numerical results. We list the results for these ten problems in Table 3. From the table we can see that simpler LGMRES almost performs better than SGMRES in terms of the three given aspects, including the number of iterations, matrix-vector multiplicatios and running time.

Table 3 A Comparison of Performance Between Simpler LGMRES and SGMRES Augmented with One Approximate Error ($k=1$).

| Problem | | SGMRES | | | simpler LGMRES | | |
|---|---|---|---|---|---|---|---|
| matrix | size(n) | iter | mv | time(sec.) | iter | mv | time(sec.) |
| add20 | 2395 | 28 | 869 | 2.9259 | 16 | 482 | 1.6950 |
| orsirr1 | 1030 | 188 | 5829 | 7.6201 | 71 | 2132 | 2.7754 |
| orsreg1 | 2205 | 17 | 528 | 1.6234 | 14 | 422 | 1.3333 |
| sherman1 | 1000 | 95 | 2946 | 3.4376 | 23 | 692 | 0.8476 |
| sherman4 | 1104 | 21 | 652 | 0.8338 | 12 | 362 | 0.4867 |
| cavity05 | 1182 | 253 | 7844 | 17.2697 | 56 | 1682 | 3.6620 |
| cavity10 | 2597 | – | – | – | 76 | 2282 | 12.6509 |
| cdde1 | 961 | 27 | 838 | 0.9578 | 19 | 572 | 0.7196 |
| e05r0000 | 236 | 114 | 3535 | 1.3493 | 23 | 692 | 0.2778 |
| nos3 | 960 | 284 | 8805 | 13.1280 | 39 | 1172 | 1.6904 |

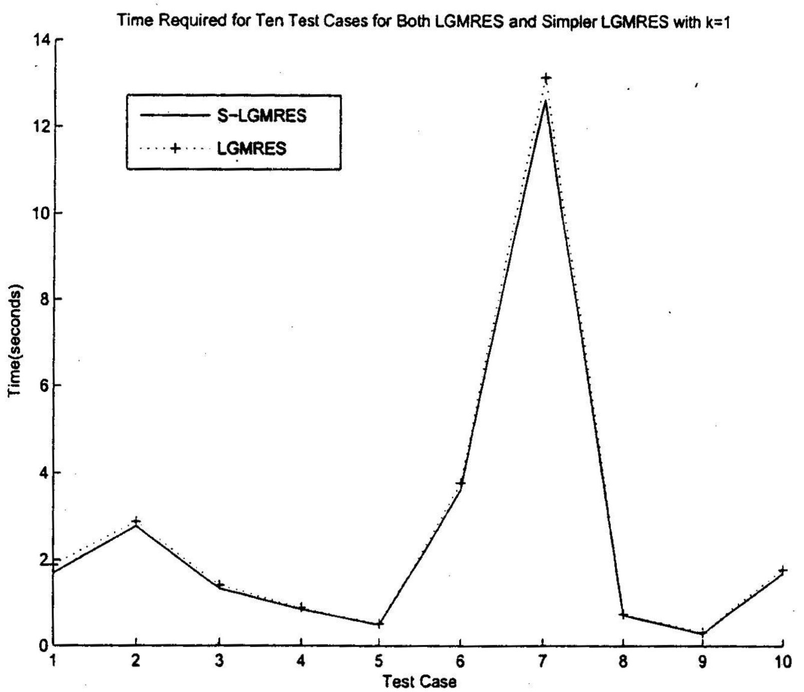


Fig. 1 A comparison of time required for a test of ten problems between simpler LGMRES and LGMRES methods with $k=1$.

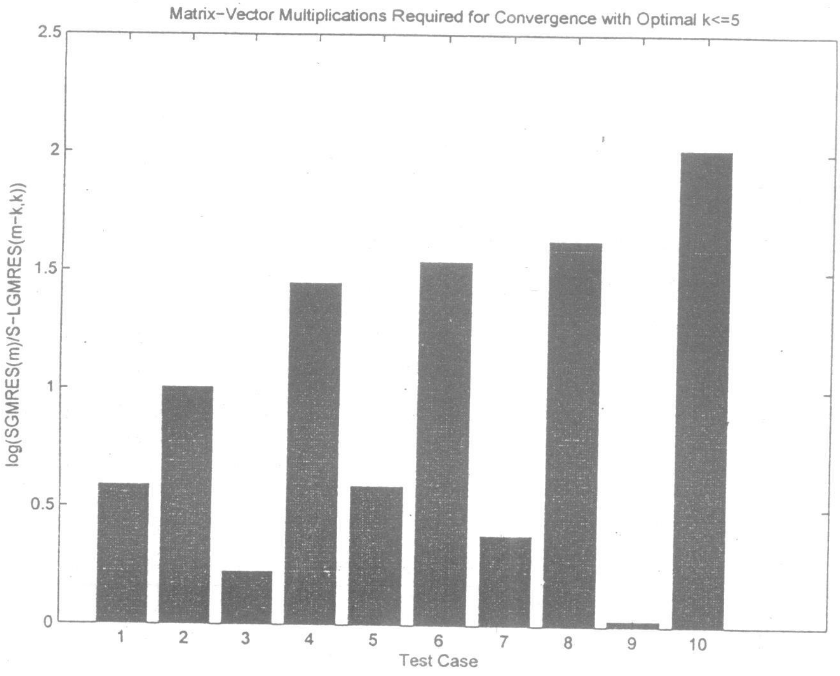


Fig. 2 A comparison of the number of matrix-vector multiplicatios required for convergence by simpler LGMRES$(m-k, k)$ and SMGRES$(m)$ for ten test case.

We also compare simpler LGMRES with LGMRES with $k=1$ to see the improvement in performance. Figure 1 shows the time required for the ten test examples using the two methods, respectively. We can see that generally it takes less time to converge within the given tolerance le-9 for simpler LGMRES than for LGMRES.

Consider the results for different values of $k$. We test the same ten test examples with $k=1:5$. We find that generally $k\leq 3$ is best for simpler LGMRES$(m-k, k)$ and $k=1$ is best for most of these problems. Figure 2 shows the number of matrix-vector multiplications required for convergence for SGMRES$(m)$ divided by the number required for simpler LGMRES$(m-k, k)$ with the optimal $k$. Note that the log of this ratio is plotted on the $y$-axis of Figure 2.

All the above numerical experiments have shown that simpler LGMRES is an effective method, superior to SGMRES and it generally performs better than LGMRES.

**Concluding remark** In this paper, we have described a simpler form of LGMRES method (simpler LGMRES) that accelerates the convergence of SGMRES. Experimental results demonstrate that simpler LGMRES$(m-k, k)$ is an effective accelerator of SGMRES$(m)$ and shows better performance than LGMRES for a wide range of problems. Furthermore, the algorithm is easy to implement, requiring a few modifications to the SGMRES implementation.

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# 添加近似误差的重新启动的 simpler GMRES 方法


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摘　要　本文给出了重新启动的 LGMRES 方法的一种代价更小的实现方式. 这种做法基于消除以下减慢收敛速度的现象: 重新启动的 simpler GMRES 的每次循环结束时得到的残向量经常交替方向, 与重新启动的 GMRES 的情形类似. 这种新的变形的方法的优点是它比重新启动的 LGMRES 所需要的计算量要少. 大量的例子表明该方法计算速度更快.



关键词　Simpler GMRES; 重新启动; Krylov 子空间方法; LGMRES