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Minimum Degree Condition for the Optimization of Restricted Edge Connectivity of Regular Graphs

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Abstract: Restricted edge cut separates a connected graph into a disconnected one without isolated vertex. Graph G is super restricted edge connected if no subgraph but an isolated edge can be separated by any minimum restricted edge cut. It is proved that k-regular connected graph G is super restricted edge connected if $k > |G|/2 + 1$. The lower bound on k is exemplified to be sharp to some extent.

Key words: graph; edge connectivity; fragment; restricted

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1 Introduction

All graphs considered in this paper are finite, simple, k-regular and connected with $k \geq 3$ if not specified. Restricted edge cut is such an edge cut that separates a connected graph into a disconnected one with no components being isolated vertices. It is trivial if its removing results in only two components with one being an edge. The cardinality of minimum restricted edge cut is restricted edge connectivity and denoted by $\lambda_2(G)$ for graph G. These concepts are referred first by Esfahanian and Hakimi[3,4] when they analyzed the fault tolerance and reliability of networks. Restricted edge connectivity proves to be a more precise measure of fault tolerance and reliability of networks than edge connectivity, and receives a lot of attention[5,6].

Denote by $I(e)$ the set of edges adjacent to edge e. Define $\xi(G) = \min\{|I(e)| : e \in E(G)\}$ to be the minimum edge degree of graph G. If G is a connected with $|G| \geq 4$ and not isomorphic to star $K_{1,n}$, then G contains restricted edge cut and $\lambda(G) \leq \lambda_2(G) \leq \xi(G)$, where $\lambda(G)$ indicates the edge connectivity. Graph G is called maximal restricted edge connected, or simply λ_2 - graph, if $\lambda_2(G) = \xi(G)$; and super restricted edge connected, or simply super λ_2-graph if furthermore G contains only trivial minimum restricted edge cuts. As was pointed out in [1], the reliability of networks corresponding to super λ_2-graphs are better than those that are not super λ_2-graphs under some reasonable conditions. Easily see that no 2-regular connected graphs but C_4 and C_5 are super λ_2-graphs. For k-regular graph with $k \geq 3$, we present the following result in this paper.

Theorem 3.2 Let G be a k-regular connected graph with. If $k > |G|/2 + 1$, then G is a super λ_2-graph.

The lower bound of k in Theorem 3.2 is exemplified to be sharp to some extent. For subset

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$X \subseteq V(G)$,we write X^c for $V(G) - X$. Denote by $[A, B]$ the set of edges of G with one end in A and the other in B ,where A, B are two disjoint subsets of $V(G)$. Sometimes $[X, X^c]$ is simplified as $I(X)$, and $[\{a\}, B]$ as $[a, B]$ for any vertex a such that $a \notin B$. Denote by $d(X)$ the cardinality of $I(X)$. Let $d_G(v)$,simply $d(v)$ if there is no confusion ,represent the degree of vertex v of G.

Let S be a minimum restricted edge-cut ,then $G - S$ has only two components ,both of which are called restricted fragments corresponding to S. We signify restricted fragment with its vertex set. It's easy to see that X^c is a restricted fragment if and only if X is a restricted fragment. Minimum restricted fragment is atom ,note that atoms have the same order ,denoted by $\mu(G)$,and contain no proper subgraph as its restricted fragment. Restricted fragment with order at most $|G|/2$ is normal restricted fragment. Obviously an atom is a normal restricted fragment. A normal restricted fragment of a λ_2-graph is called a quasi-atom if it has order at least 3 and contains no restricted fragments but K_2. A λ_2-graph is super restricted edge connected if and only if it contains no quasi-atoms. It is worth noting that restricted fragments are connected vertex induced subgraphs ,which appear in pair.

Let $\delta(G)$ indicate the minimum degree of graph G. The joint of graph G and L is a graph H with $V(H) = V(G) \cup V(L)$, $E(H) = E(G) \cup E(L) \cup E_1$,where $E_1 = \{uv : u \in V(G), v \in V(L)\}$. Other symbols and terminology not explicitly stated coincide with that in [2].

2 Auxiliaries

Before presenting the main results ,we introduce some properties of restricted fragment. On the one hand ,these properties are of their own importance. On the other hand ,they are the basis of the main result.

Lemma 2.1 Let X and Y be two different restricted fragments of graph G. If the following two conditions hold ,then $X \cap Y$ is a restricted fragment.

(1) $|X \cap Y| \geq 2$, (2) $d(X \cap Y) \leq \lambda_2(G)$.

Proof Define

$$A = X \cap Y, \quad B = X \cap Y^c, \quad C = X^c \cap Y, \quad D = X^c \cap Y^c$$

It suffices to verify that both A and A^c are connected.

(a) $A^c = X^c \cup Y^c$ is connected.

Since both X^c and Y^c are all connected fragments ,Assertion (a) is true in the case when D is not empty. If D is empty ,then $Y^c = B$ and $X^c = C$. We thus need only to show that [B ,C] is not empty. Suppose , to the contrary ,that such is not the case. Then

$$\lambda_2(G) \geq d(X \cap Y) = d(A) = |[A, B]| + |[A, C]| = 2\lambda_2(G)$$

This contradiction to Condition (2) establishes (a).

(b) $A = X \cap Y$ is connected.

Otherwise ,there are at least two components in A. If all of these components are isolated vertices ,then

$$d(A) = \sum_{v \in A} d(v) \geq 2\delta(G) > 2k - 2 = \xi(G) \geq \lambda_2(G)$$

This contradiction implies that there is at least one component A_1 in A with order at least 2. By (a) , $G \setminus A_1$ is connected. Hence $I(A_1)$ is a restricted edge cut ,but

$$\lambda_2(G) \leq d(A_1) < d(A) < d(X) = \lambda_2(G)$$

Which is a contradiction.

Lemma 2.2 Let X and Y be two different normal restricted fragments of G. If $|X \cap Y| \geq 2$,then both $X \cap Y$ and $X^c \cap Y^c$ are restricted fragments.

Proof Define A, B, C and D the same as in Lemma 2.1. We shall establish the lemma by the following four claims.

(1) $|D| \geq |A| \geq 2$.

Since X and Y are two normal restricted fragments, it follows that $|X| \leq \frac{|G|}{2} \leq |X^c|$ and $|Y| \leq \frac{|G|}{2} \leq |Y^c|$, Therefore

$$|A| + |C| = |Y| \leq \frac{|G|}{2} \leq |X^c| = |C| + |D|.$$

Statement (1) is thus true.

(2) $|[A, B]| \leq [D, B]|$.

Suppose by contradiction that $|[A, B]| > |[D, B]|$. Then

$$d(D) = |[D, Y]| + |[D, B]| < |[D, Y]| + |[A, B]| + |[B, C]| = \lambda_2(G) \qquad \text{(a)}$$

Combining this observation with Claim (1), we deduce from Lemma 2.1 that D is a restricted fragment. Hence $d(D) = \lambda_2(G)$, contradicting Formula (a).

(3) $A = X \cap Y$ is a restricted fragment.

By (2), we have

$$d(A) = |[A, X^c]| + |[A, B]| \leq |[A, X^c]| + |[D, B]| + |[B, C]| = \lambda_2(G)$$

Combining this observation with Claim (1) and Lemma 2.1, we derive the desired result.

(4) $D = X^c \cap Y^c$ is a restricted fragment.

By Claim (3), we have

$$\lambda_2(G) + d(D) = d(A) + d(D) = d(X) + d(Y) - 2|[B, C]| \leq 2\lambda_2(G)$$

Therefore $d(D) \leq \lambda_2(G)$. The statement follows from the combination of this observation with Claim (1) and Lemma 2.1. Our proof thus finishes.

Corollary 2.3 Let X and Y be two distinct atoms of G. Then $|X \cap Y| \leq 1$.

Proof Note that atoms are normal restricted fragments. Combining this observation with Lemma 2.2, we conclude that $X \cap Y$ is a restricted fragment if $|X \cap Y| \geq 2$. But this is impossible since X and Y are atoms that contain no proper subgraphs as their fragments.

Lemma 2.4 Let X and Y be two different atoms. If $X \cap Y$ is not empty, then both X and Y are isomorphic to K_2.

Proof Suppose by contradiction that X and Y are two different atoms such that $X \cap Y$ is not empty, but at most one of X and Y is isomorphic to K_2. Then $|X| = |Y| \geq 3$. Define A, B, C and D as before. From Lemma 2.3, $X \cap Y$ has only one element, say vertex a, which implies that

$$|B| = |C| \geq |Y| - 1 \geq 3 - 1 = 2.$$

Since either $|[a, C]| \geq [a, B]$ or $|[a, C]| \leq |[a, B]|$, there is no loss of generality in assuming that the former is true. It follows directly from this assumption that $d(X) \geq d(B)$. By Lemma 2.1, B is a restricted fragment, which contradicts the condition that X is an atom.

Corollary 2.5 Let G be a k-regular connected graph with $k \geq 2$ and $|G| \geq 4$.

(1) All atoms of graph G are K_2 or

(2) None of the atoms of G is K_2 and the intersection of any two atoms is empty.

Proof It is easy to check the case when $k = 2$. By Lemma 2.4, Corollary 2.5 is also true in the case when $k \geq 3$.

Lemma 2.6 The following statements are equivalent.

(1) Graph G is not λ_2-graph.

(2) $\mu(G) \geq k$.

(3) Atoms of G are disjoint.

Proof (1) ⇒ (2) Let X be an atom. Since G is not λ_2-graph, it follows that

$$2k - 2 > \lambda_2(G) = d(X) = k|X| - \sum_{v \in X} d_X(v) \geq k|X| - |X|(|X| - 1)$$

$$(|X| - 2)(|X| - k + 1) > 0.$$

By Corollary 2.5, we have $|X| > 2$, hence $|X| > k - 1$. Since $|X|$ is an integer, it follows that $\mu(G) = |X| \geq k$.

(2) ⇒ (3) Let X and Y be two distinct atoms of G. Since $\mu(G) \geq k > 2$, $X \cap Y$ must be empty by Lemma 2.4. Otherwise, $X \cap Y$ or $X \cap Y^c$ is a fragment contained in X, a contradiction.

(3) ⇒ (1) Claim at first that each edge of λ_2-graph G is an atom. In order to confirm this claim, we need only to show that $I(e)$ constitutes a restricted edge-cut of G for any edge e, namely $G - I(e)$ has no isolated vertices. But this is apparently since $k > 2$. Now, it follows directly from this claim that there exist two different atoms whose intersection is not empty if G is a λ_2-graph, a contradiction.

Corollary 2.7 Let G be a k-regular connected graph with $k \geq 1$ and $|G| \geq 3$. If $k > |G|/2$, then G is a λ_2-graph.

Proof If G is not λ_2-graph, then $\mu(G) \geq k > |G|/2$ by Lemma 2.6. This contradiction completes our proof.

3 Main result and its proof

Lemma 3.1 Let G be a λ_2-graph. If X is a quasi-atom of G, then $|X| \geq k - 1$. The equality holds if and only if X is a $(k-1)$-clique.

Proof Since G is a λ_2-graph, X is a quasi-atom, it follows that

$$2k - 2 = d(X) = k|X| - \sum_{v \in X} d_X(v) \geq k|X| - |X|(|X| - 1),$$

$$|X|^2 - (k+1)|X| + 2(k-1) \geq 0,$$

$$(|X| - 2)(|X| - k + 1) \geq 0.$$

According to the definition of quasi-atom, we have $|X| - 2 > 0$. Therefore $|X| - k + 1 \geq 0$, furthermore, $|X| - k + 1 = 0$ if and only if all the equalities in the previous inequalities hold, namely if and only if $\sum_{v \in X} d_X(v) = |X|(|X| - 1)$. This implies that $|X| - k + 1 = 0$ if and only if X is a $(k-1)$-clique.

Theorem 3.2 Let G be a k-regular connected graph. If $k > |G|/2 + 1$, then G is a super λ_2-graph.

Proof Through Corollary 2.7, G is a λ_2-graph. If G is not super restricted edge connected, then there is a quasi-atom X of G with the property that $|G|/2 \geq |X| \geq k - 1 > |G|/2 + 1 - 1 = |G|/2$ by Lemma 3.1, which is a contradiction.

Remark The following graph G exemplifies that the lower bound on k in Theorem 3.2 cannot be improved to some extent.

Let H_1 and H_2 be two graphs isomorphic to K_m, the complete graph on m vertices, $V(H_i) = (v_1^i, \cdots, v_m^i)$, $i = 1, 2$, where m is an integer satisfying $m > 2$. Adding edge set $E = \{v_i^1 v_i^2, v_i^1 v_{i+1}^2 : i = 1, \cdots, m-1\} \cup \{v_m^1 v_m^2, v_m^1 v_1^2\}$ between H_1 and H_2 leads to our desired graph G. It's easy to see that

(1) G is k-regular, $k = m + 1$.

(2) $k = |G|/2 + 1$.

(3) E is a minimum restricted edge-cut of G.

(4) H_1 and H_2 are quasi-atoms of G.

Since a λ_2-graph G is a super λ_2-graph if and only if there are no quasi-atoms in G, G is not a super λ_2-graph.

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优化正则图的限制边连通性的最小度条件

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摘　要: 限制边割将连通图分离成不含孤立点的不连通图 ,如果最小限制边割只能分离孤立边 ,则称图 G 是超级限制边连通的. 证明了如果 $k > |G| / 2 + 1$,那么 k 正则连通图 G 是超级限制边连通的 , k 的下界在一定程度上是不可改进的.

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一个非线性扩散方程的 PAINLEVÉ - BÄCKLUND 变换及其精确解

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摘　要: 选择 Painlevé- Bäcklund 方程组的不同解 ,给出一类非线性扩散方程的某些精确孤立波解. 这个方法也可以用来寻找其他非线性偏微分方程的精确孤立波解.

关键词: 非线性扩散方程 ; Painlevé- Bäcklund 方程组 ; 孤立波解

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