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Transport Properties of a Classical One-Dimensional Kicked Billiard Model

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We study a classical 1-dimensional kicked billiard model and investigate its transport behavior. The roles played by the two system parameters $\alpha$ and $K$, governing the direction and strength of the kick, respectively, are found to be quite crucial. For the perturbations which are not strong, i.e. $K < 1$, we find that as the phase parameter $\alpha$ changes within its range of interest from $-\pi/2$ to $\pi/2$, the phase space is in turn characterized by the structure of a prevalently connected stochastic web ($-\pi/2 \leq \alpha < 0$), local stochastic webs surrounded by a stochastic sea ($0 < \alpha < \pi/2$) and the global stochastic sea ($\alpha = \pi/2$). Extensive numerical investigations also indicate that the system's transport behavior in the irregular regions of the phase space for $K < 1$ has a dependence on the system parameters and the transport coefficient $D$ can be expressed as $D \approx D_0(\alpha)K^\beta(\alpha)$. For strong kicks, i.e. $K \gg 1$, the phase space is occupied by the stochastic sea, and the transport behavior of the system seems to be similar to that of the kicked rotor and independent of $\alpha$.

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In recent years, discontinuous dynamical systems have been attracting increasing interest.1-5 As is well known, when discontinuity is introduced into a dynamical system, many unusual phenomena may appear, some of which can be of great physical interest, e.g. for a 2-dimensional (2-D) discontinuous map whose phase space is a plane or a cylinder, the KAM theorem is not valid due to the discontinuity and hence the motion of the system is typically unbounded no matter how weak the outside perturbations are. This is quite different from the behavior of those continuous systems, e.g. the Chirikov standard map (CSM), whose motions are bounded by the KAM invariant torus when the perturbation strength is below a certain threshold.6

As a consequence, although the transport properties of 2-D continuous maps are now very well understood through the study of some prototype models, e.g. CSM, our knowledge of how the discontinuity would influence the transport properties of the discontinuous 2-D map systems is still limited. In this paper, we shall study a 1-dimensional kicked billiard model (1D-KBM) and observe how its diffusion behavior changes as the system parameters are continuously varied.

Consider a particle of unit mass which is constrained to move frictionlessly along a unit circle and is subjected to a vertical periodic impulsive force of impulse strength $K$ applied at integral times. Its dynamics is easy to derive, and turns out to be the well known CSM. If we further constrain its motion by setting two rigid boundaries at two opposite positions on the circle, i.e. two ends of a diameter, then we can achieve a 1D-KBM, whose Hamiltonian is

Here $\theta$ and $I$ denote the position and angular momentum of the particle, respectively; $\delta_T$ is a $\delta$-function sequence with period $T$, and $\alpha$ is introduced as another system parameter, represents the angle between the vertical direction (at which the impulse is applied) and the diameter that connects the two positions of the boundaries (note that we have reset the angular coordinate to put them at $\theta = 0$ and $\theta = \pi$). This model was first proposed by one of the authors (GU Yan) in another paper where the general properties of its quantum counterpart were presented.7

The phase space of this system is $[0, \pi] \times R$. After each bounce at boundaries, the motion direction of the billiard ball should be changed. This makes it somewhat trivial in tracing its motion. In order to get rid of this inconvenience, we can consider the following Hamiltonian with a cylinder phase space instead:

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$H = \frac{I^2}{2} + K \cos(\theta - \alpha)\delta_T(t) + v(\theta),$

$v(\theta) = \begin{cases} 0, & 0 \leq \theta \leq \pi, \\ +\infty, & \pi < \theta < 2\pi. \end{cases}$
characterized by a local stochastic web centered at

and surrounded by the stochastic sea if $4\alpha + K < 2\pi$

Fig. 1. Phase portrait of the 1D-KBM for (a) $\alpha = 0.4$, $K = 0.2$ and (b) $\alpha = -0.4$, $K = 0.2$.

From the Hamiltonian (2), it is straightforward to get the canonical kinetic equations. By integrating them from just before the $n$-th kick to just before the $(n+1)$-th kick, we have

$$\begin{cases}
I_{n+1} = I_n + K \sin[\theta_n - S(\theta_n)\alpha], \\
\theta_{n+1} = \theta_n + I_{n+1}, \quad (\text{mod} \ 2\pi).
\end{cases}$$

(3)

How the system’s behavior, in particular diffusion behavior depends on the parameter $\alpha$, is the most interesting problem for us. Due to the symmetry of the map we have $\theta \rightarrow \pi - \theta (\text{mod} \ 2\pi)$, $I \rightarrow -I$ and $\alpha \rightarrow \pi - \alpha (\text{mod} \ 2\pi)$, so the scope of parameter $\alpha$ that needs to be investigated is just $[-\pi/2, \pi/2]$. In fact, this map has been studied at two individual values of $\alpha$. One is $\alpha = 0$, at which the system degenerates to the well studied continuous CSM, and another is $\alpha = \pi/2$, at which the system has a global chaotic phase space and the diffusion rate has a piecewise linear dependence on the parameter $K$. The results obtained from these two systems can be used as references for our aim here.

Numerical investigations show that the map (3) has distinct phase space structures for two different value ranges of $\alpha$, i.e. $(0, \pi/2]$ and $[-\pi/2, 0)$. In Fig. 1 the phase portrait for two representative values of $\alpha$ are presented. Since the phase portrait is of period $2\pi$ in $I$, the range of $I$ shown in Fig. 1 can be restricted to $[-\pi, \pi]$.

For $0 < \alpha < \pi/2$, the phase portrait is characterized by a local stochastic web centered at $(\pi, 0)$ and surrounded by the stochastic sea if $4\alpha + K < 2\pi$.

[Fig. (1a)]. This makes it quite different from both the infinite stochastic web studied by Schwägerl and Krug and that studied by Zaslavsky and coworkers. The meshes of the web are organized in tori of different radii, and each torus is recognized as an island chain of a certain stable periodic orbit. For a given parameter $K$ that is less than $2\pi - 4\alpha$, we find that as $\alpha$ is increased the islands of the outmost island chain will shrink monotonically and vanish above a certain value of $\alpha$ at which the embedded periodic orbit begins to lose its stability. During this process the other inner island chains may just expand outward a little, but in general the size of the local stochastic web decreases. As $\alpha$ is increased further, the same thing will happen to the next island chains successively till $\alpha$ reaches $(2\pi - K)/4$, at which the last island chain (of period 2) dies away. Henceforth the phase space is taken up by the stochastic sea completely. On the other hand, for a given value of $\alpha$ in $(0, \pi/2)$ that is less than $(2\pi - K)/4$, almost the same evolution of the phase portrait can be observed as $K$ increases, except that the size of the local stochastic web may change in a more complicated way.

Now let us turn to the other parameter region $\alpha < 0$. In this case, if $K$ is not too large, then we shall find that the phase space is occupied by a prevalently connected stochastic web and the inlaid island chains of stable periodic orbits [Fig. (1b)]. The system now has two fixed points at $(\pi + \alpha, 0)$ and $(\pi - \alpha, 0)$, and around their islands there is an annulus structure. If $K$ is kept unchanged while we vary $\alpha$ from $0$ to $-\pi/2$, these two fixed points will separate from each other gradually and their islands will keep broadening till they finally connect at $\theta = 0$. During this process, the width of the annulus structure decreases, and with the successive appearance of new island chains and the disappearance of existing ones, the phase space structure may change greatly. Finally, at $\alpha = -\pi/2$, the stochastic web turns into a more symmetrical pattern that looks as if it has a period of $\pi$ in the $I$ direction. On the other hand, if we increase $K$ from just above 0 but keep $\alpha$ unchanged, we shall find that every island chains that may appear in the phase space will experience a life period from being born at a certain value of $K$ to death at another larger $K$. So it is not easy to describe in detail how the phase space develops as $K$ increases, but generally speaking, the number of the island chains will decrease and the structure of the stochastic web will become more and more simple.

These two types of phase space structure encountered here are new to us. Numerical investigations indicate that in the irregular parts of the phase space the system will transport along the angular momentum direction according to $\langle I_n^2 \rangle = D_n$ for $n \rightarrow \infty$. Here $n$ is the time measured in iterations of the map (3), the average $\langle \cdot \cdot \cdot \rangle$ is taken relative to some ensemble of initial conditions and $D$ is the global diffusion constant. In Fig. 2 the dependence of $D$ on $K$ for several typical values of $\alpha$ are illustrated. The ensemble used in our calculations contains 1000 initial points...
whose angular coordinates are set as $\pi$ and the angular momentum are set randomly in a small range around 0. From this figure it can be found that $D$ has a different power law dependence on $K$ for $K < 1$ and $K \gg 1$, respectively. For $K < 1$, detailed numerical calculations suggest that

$$D(\alpha, K) = D_0(\alpha)K^{f(\alpha)}, \quad (K < 1) \quad (4)$$

where $D_0$ and $f$ can be regarded as two functions of $\alpha$. As shown in Fig. 3(a), $f(\alpha)$ is about 3/2 for $\alpha = 0_+$ and $\alpha = 0_-$, and after a rapid increase it climbs up to 5/2 and stays there as the absolute value of $\alpha$ increases. $D_0(\alpha)$ is illustrated in Fig. 3(b), where it has a dependence on $\alpha$ that can be approximated as $D_0(\alpha) \approx 0.4 \sin^2(\alpha)$ for $0 < \alpha < \pi/2$ and $D_0(\alpha) \approx 0.038 \sin^2(\alpha)$ for $-\pi/2 < \alpha < 0$. These two functions provide us a complete picture on how the system’s diffusion depends on the parameter $\alpha$ for $K < 1$. As for $K \gg 1$, it seems that $D$ has nothing to do with $\alpha$ (Fig. 2). We can derive from our numerical data that

$$D(\alpha, K) = \frac{1}{2} K^2 \quad (K \gg 1). \quad (5)$$

In fact, by using a quasilinear phase approximation, i.e. assuming the sequences of phase $\{\theta_n\}$ produced by iterating map (3) to be completely random irrelative variables for large $K$, we can deduce this expression easily. Therefore it is not difficult to understand why $D$ is independent of $\alpha$ for $K \gg 1$.

We note that there is an interesting phenomenon in the slow diffusion case $K < 1$. Although the phase space structures for two parameter regions of $\alpha$ are distinctly different, the scaling exponent $f(\alpha)$ exhibits similar dependence on $\alpha$ in these two regions. In addition, we also find that this is a generic phenomenon for various versions of 1D-KBM. As an example, we present also in Fig. 3(a) the dependence of $f(\alpha)$ on $\alpha$ that is derived from studies on the following piecewise linear 1D-KBM:

$$I_{n+1} = \begin{cases} K(\theta_n - \alpha), & (0 \leq \theta_n < \alpha + \pi/2), \\ K(-\theta_n + \alpha + \pi), & (\alpha + \pi/2 \leq \theta_n < \pi), \\ K(-\theta_n - \alpha + \pi), & (\pi \leq \theta_n < 3\pi/2 - \alpha), \\ K(\theta_n + \alpha - 2\pi), & (3\pi/2 - \alpha \leq \theta_n < 2\pi), \end{cases}$$

$$\theta_{n+1} = \theta_n + I_{n+1}, \quad (\text{mod} 2\pi). \quad (6)$$

For $\alpha = \pi/2$, Eq. (6) gives a sawtooth map that has been used as an example to illustrate the results of a transport theory based on the partition of phase space into resonance. This theory provides an explanation of the scaling exponent $5/2$ in this special case, but it can hardly be extended to generic discontinuous systems. In order to understand the significance of formula (4), new transport theory must be developed.

In conclusion we have studied 1D-KBM. Due to the discontinuity of its dynamics, the system can slowly diffuse in phase space no matter how weak the outside perturbations are. For weak perturbations ($K < 1$), the system’s diffusion rate has a power law dependence on $K$ with the exponent being decided by another system parameter concerning the discontinuity, while for strong perturbations ($K \gg 1$) it seems that the influence of the discontinuity on the system’s diffusion behavior can be neglected.

REFERENCES