Electroelastic analysis of a piezoelectric half-plane with finite surface electrodes

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Abstract

In this paper the problem of the electroelastic fields of piezoelectric ceramic in a half-plane with thin collinear surface electrodes is formulated and analyzed. Using the standard analytic continuation method, a mixed boundary value problem in plane is obtained and then reduced to a standard Hilbert problem. The closed form expressions of electroelastic fields are obtained. Some special cases are discussed. The electroelastic filed has singularity with $1/\sqrt{r}$ at the vicinity of the electrode ends. The numerical examples for two surface electrodes are studied to show the practical distributions of the electroelastic fields graphically. © 2004 Elsevier Ltd. All rights reserved.

Keywords: Piezoelectric ceramic; Surface electrodes; Electroelastic behavior; Hilbert problem

1. Introduction

Owing to the large coupling between electric and mechanical fields, piezoelectric ceramics are widely used as smart materials such as transducers, sensors and actuators. In these cases, the thin electrodes are usually attached at the surface or interface of electroceramic blocks [1–8]. Hence it is important to research the electroelastic field produced by the thin electrodes in piezoelectric apparatuses. Shindo et al. [2] studied the static behavior of electroelastic field at the vicinity of single surface electrode by Fourier transformation method in a transverse isotropic material. Ru [3] studied the case of thin electrodes embedded at the interface of two bonded dissimilar
piezoelectric half-planes. Ye and He [6] analyzed the electric field concentrations of the edges of a pair of parallel electrodes in a transverse isotropic material when the sizes of electrodes are far smaller than that of the thickness. They found the stresses at the edge of electrodes are very high and can make the ceramics failure. Li and Tang [7] studied the piezoelectric layer with electrodes at the top surface and bottom surface by the conformal mapping technique. Bent and Hagood [8] discussed in detail the properties of a piezoelectric fiber composites with interdigitated electrodes. Zhou et al. studied a single surface electrode for soft and rigid cases [9] by conformal mapping method and got pretty closed solutions.

In this paper, the problem of thin collinear electrodes attached at the surface of the half-plane in a general piezoelectric ceramic are formulated and discussed. According to the principle of standard analytic continuation method, the problem is continued to the whole plane and then is reduced to a mixed boundary problem which can be reduced to a Hilbert problem. The closed form solutions of electroelastic filed are obtained, which show the electric and stress fields exhibit inverse square root singularity at the vicinity of the electrode ends. The field concentration is abated when the depth \((x_2/C_0)\) increases. Finally some special examples are given. The distributions of the electric field and stress are useful in engineering application.

2. Basic equations and general solution

In a fixed rectangular coordinate system \((x_1, x_2, x_3)\), all of the field variables depend on \(x_1, x_2\) only for a generalized plane piezoelectric problem. Following Suo et al. [10] and Kuang and Ma [11], the general solution in this case can be given by the linear combination of four complex analytical functions

\[
\begin{align*}
\mathbf{u} &= 2\text{Re}[\mathbf{Af}(z)], \quad \phi = 2\text{Re}[\mathbf{Bf}(z)], \\
\mathbf{u} &= [u_1, u_2, u_3, \varphi]^T, \quad \phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T, \\
\mathbf{f}(z) &= [f_1(z_1), f_2(z_2), f_3(z_3), f_4(z_4)]^T, \quad z_k = x_1 + p_k x_2 \quad (k = 1-4),
\end{align*}
\]

where Re stands for the real part of a function; \(u_i\) \((i = 1, 2, 3)\), \(\varphi\) and \(\phi_i\) are the displacement components, electric potential and the components of the generalized stress function, respectively; \(\mathbf{A}\) and \(\mathbf{B}\) are \(4 \times 4\) complex matrices related to the material constants, expressed as

\[
\mathbf{A} = [a_1, a_2, a_3, a_4], \quad \mathbf{B} = [b_1, b_2, b_3, b_4].
\]

The eigenvalues \(p_k\) and eigenvectors \(a_k\) are determined by the following equations:

\[
[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)p + \mathbf{T}p^2]a = 0
\]

where

\[
\begin{align*}
\mathbf{Q} &= \begin{bmatrix} \mathbf{Q}_{11}^E & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\kappa_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{11}^E & \mathbf{e}_{21} \\ \mathbf{e}_{21}^T & -\kappa_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}_{11}^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\kappa_{22} \end{bmatrix}, \\
Q_{1k}^E &= c_{1k1}, \quad R_{1k}^E = c_{1k2}, \quad T_{1k}^E = c_{1k2}, \quad (\mathbf{e}_{ij})_s = e_{ij},
\end{align*}
\]
where $c_{ijkl}$ are the elastic stiffnesses under constant electric field, $e_{ij}$ are the piezoelectric constants and $\kappa_{ij}$ are the permittivities under constant strain field. The eigenvectors $b_k$ can be obtained from the following equations:

$$b_k = (R^T + p_k^T)A_k = -\frac{1}{p_k}(Q + p_kR)A_k.$$  \hspace{1cm} (5)

The generalized stresses are represented as

$$\mathbf{t}_1 = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1]^T = -[\phi_{1,2}, \phi_{2,2}, \phi_{3,2}, \phi_{4,2}]^T;$$

$$\mathbf{t}_2 = [\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2]^T = [\phi_{1,1}, \phi_{2,1}, \phi_{3,1}, \phi_{4,1}]^T,$$  \hspace{1cm} (6)

where $\sigma_{ij}, D_i$ are stresses and electric displacements, respectively.

After the normalization for the eigenvectors $A$ and $B$, there exist the following relations [11,12]:

$$\begin{bmatrix} B^T \\ A^T \end{bmatrix} \begin{bmatrix} A & \bar{A} \\ B & \bar{B} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} A & \bar{A} \\ B & \bar{B} \end{bmatrix} \begin{bmatrix} B^T \\ A^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$  \hspace{1cm} (7)

3. Statement and solution procedure of the problem

Consider thin collinear electrodes attached at the surface of piezoelectric ceramic as shown in Fig. 1. Let the left and right ends of the $i$-th electrode be $a_i$ and $b_i$ ($i = 1, 2, \ldots, n$), respectively. It is assumed that the total electric charge or potential on the $i$-th electrode is $q_i$ or $V_i$, respectively. Then, the mixed boundary conditions are given by

Fig. 1. A piezoelectric ceramic with collinear surface electrodes.
\[ \sigma_{ij} = 0, \quad D_j = 0; \quad x_2 = 0, \quad z \in L' = [-\infty, a_1] + [b_1, a_2] + \ldots + [b_n, +\infty], \]
\[ \sigma_{ij} = 0, \quad E_i = 0; \quad x_2 = 0, \quad z \in L'' = [a_1, b_1] + \ldots + [a_n, b_n], \]
\[ \sigma_{ij} \to 0, \quad D_j \to 0; \quad |z| \to \infty, \quad i = 1, 2, 3, \quad j = 1, 2, \]

where \( L' + L'' = L \). Analogous to elastic contact problem [13], using the Gauss law, along the surface electrodes the supplement boundary conditions can be written as

\[ \int_{L''} D_2^2 - (x_1) dx_1 = -q_i, \quad i = 1, 2, \ldots, n \tag{9a} \]

or

\[ \phi_i = V_i, \quad i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \int_{L''} D_2^2(x_1) dx_1 = -Q = -\sum_{i=1}^{n} q_i, \tag{9b} \]

where \( Q \) is the total charge on all electrodes. If \( Q = 0 \), then the supplement conditions can be expressed by pure potentials. If \( Q \neq 0 \), then \( Q \) is included in the solution. Eqs. (9a) and (9b) are equivalent. The superscript ‘-’ indicates the limit value taken from the lower half-plane. The boundary conditions (8) and (9a) or (9b) guarantee that the discussed problem has an unique solution. Let \( \mathbf{F}(z) = \mathbf{f}(z) \), following Eq. (8)_i, the generalized stresses along all electrode-free part \( L' \) can be written as

\[ \mathbf{BF}^{-}(x_1) + \mathbf{BF}^{+}(x_1) = 0, \quad x \in L', \tag{10} \]

which can be rewritten as [13]

\[ \mathbf{BF}^{-}(x_1) + \mathbf{BF}^{+}(x_1) = 0, \tag{11} \]

where the superscript ‘+’ indicates the limit value taken from the upper half-plane. Now a new function of complex variable \( z \) is defined as

\[ h(z) = \begin{cases} -\mathbf{B}^{-1}\mathbf{BF}(z) & z \in S^+ \\ \mathbf{F}(z) & z \in S^- \end{cases} \tag{12} \]

The function \( h(z) \) is analytic in the whole plane except at the segment \( L'' \). Using Eqs. (1) and (6), Eq. (8)_2 can be written as

\[ \mathbf{BF}^{-}(x_1) + \mathbf{BF}^{+}(x_1) = \mathbf{g}, \quad \mathbf{g} = (0, 0, 0, D_2^2(x_1))^T, \quad x \in L'', \tag{13} \]

where \( D_2^2(x_1) \) is unknown. Substituting Eq. (12) into Eq. (13), we get

\[ h^+(x_1) - h^-(x_1) = -\mathbf{B}^{-1}\mathbf{g}, \]
\[ h^x_+(x_1) - h^x_-(x_1) = -B^{-1}_x D_2^x(x_1), \quad x = 1, \ldots, 4, \tag{14} \]
which is the standard Hilbert problem. The solution of Eq. (14) is

\[
\mathbf{h}(z) = -\frac{\mathbf{B}^{-1}}{2\pi i} \int_{C_0} \frac{\mathbf{g}(x_1)}{x_1 - z} \, dx_1 \quad \text{and} \quad F_a(z_x) = h_x(z_x) \quad \text{when} \quad z \in S^-.
\]

(15)

It is known that the electric displacement can be expressed in the form as [13,14]

\[
D_2(z) = \frac{P(z)}{\prod_{i=1}^{n} \sqrt{(z - a_i)(z - b_i)}}, \quad z \in S^-, \quad (16)
\]

where \(P(z)\) is a polynomial of degree \((n - 1)\) and we select the Branch \(\sqrt{(z - a_i)(z - b_i)} \to x\) when \(z \to \infty\) along axis \(x\),

\[
P(z) = i(\gamma_{n-1}z^{n-1} + \cdots + \gamma_1z + \gamma_0) \quad (17)
\]

in which \(\gamma_i (i = 1, 2, \ldots, n - 1)\) are \(n\) unknown complex numbers which should be determined by Eq. (9a) or (9b).

Substituting Eqs. (16) and (17) into Eq. (15) and noting that \(D\) takes its value at \(S^-\), we get [13]

\[
F_a(z_x) = B_{a-1}^{-\frac{1}{2}} \frac{P(z_x)}{\prod_{i=1}^{n} \sqrt{(z_x - a_i)(z_x - b_i)}}, \quad (18)
\]

\[
f_a(z_x) = B_{a-1}^{-\frac{1}{2}} \int \frac{P(z_x)dz_x}{\prod_{i=1}^{n} \sqrt{(z_x - a_i)(z_x - b_i)}} + \frac{iC}{2} B_{a-1}^{-\frac{1}{2}}, \quad (19)
\]

where \(C\) can be taken as a real constant due to that its imaginary part does not infect the potential (see Eq. (27)). The expression of \(f_a(z_x)\) may not be expressed by elementary functions except some special cases, so in some cases using Eq. (9a) is more convenient. Using \(E_1 = 0\) on \(L''\) shown in Eq. (8), we have

\[
\mathbf{A} \mathbf{F}(x_1) + \overline{\mathbf{A} \mathbf{F}(x_1)} = (*, *, *, 0) \quad (20)
\]

in which "*" denotes some unknown quantities which are not concerned here. Eq. (20) can be rewritten as
\( A_{4\varepsilon} B_{\varepsilon}^{-1} \frac{iP(x_1)}{\sqrt{(x_1 - a_i)(b_i - x_1)} \prod_{j=1, j \neq i}^{n} \sqrt{(x_1 - a_j)(x_1 - b_j)}} \)

\[- A_{4\varepsilon} B_{\varepsilon}^{-1} \frac{iP(x_1)}{\sqrt{(x_1 - a_i)(b_i - x_1)} \prod_{j=1, j \neq i}^{n} \sqrt{(x_1 - a_j)(x_1 - b_j)}} = 0, \quad x_1 \in L''_i, \quad (21)\]

where \( i = \sqrt{-1} \). According to [3], [13] and [14] and that \( H_{44} = iA_{4\varepsilon} B_{\varepsilon}^{-1} \) is real, so Eq. (21) can be reduced to

\[ P(x_1) + P(x_1) = 0, \quad x_1 \in L''_i. \quad (22)\]

So we can get the relation of \( \gamma_i \)

\[ \gamma_i - \overline{\gamma_i} = 0, \quad (23)\]

which means that \( \gamma_i \) are all the real constants.

The generalized stresses can be expressed as

\[ t_{2\beta} = \text{Re} \sum_{x=1}^{4} B_{\beta x} \frac{P(z_\beta)}{\prod_{i=1}^{n} \sqrt{(z_\beta - a_i)(z_\beta - b_i)}} B_{\varepsilon}^{-1}. \quad (24)\]

The generalized displacements are

\[ u_{\beta} = 2\text{Re}[A_{\beta x} f_x(z_\beta)]. \quad (25)\]

It is obvious that \( t_{2\beta} \) exhibit inverse square root singularity at the vicinity of electrode ends. Substituting Eq. (24) into (9a) we get

\[ \int_{a_i}^{b_i} t_{24}(x_1)dx_1 = \int_{a_i}^{b_i} D_2(x_1)dx_1 = \int_{a_i}^{b_i} \text{Re} \frac{P(x_1)dx_1}{\prod_{j=1}^{n} \sqrt{(x_1 - a_j)(x_1 - b_j)}} \]

\[ = \int_{a_i}^{b_i} \frac{iP(x_1)dx_1}{\prod_{j=1, j \neq i}^{n} \sqrt{(x_1 - a_j)(b_i - x_1)(x_1 - a_j)(x_1 - b_j)}} = -q_i. \quad (26)\]

There are \( n \) equations to determine \( n \) unknowns \( \gamma_0, \gamma_1, \ldots, \gamma_{n-1} \), so the problem is solved. Or substituting Eq. (25) into (9b) we get
\[(u_{44}) = \varphi_i = H_{44} \text{Re} \int_{a_i}^{b_i} \frac{-iP(x_i)dx_i}{\prod_{j=1}^{n}(x_i - a_j)(x_i - b_j)} + H_{44}C = V_i,\]

\[\sum_{i=1}^{n} \int_{a_i}^{b_i} \text{Re} \frac{P(x_i)dx_i}{\prod_{i=1}^{n}(x_i - a_i)(x_i - b_i)} = -Q.\]  

\(27\)

There are \(n + 1\) equations to determine the \(n + 1\) unknowns \(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\) and \(C\). It is also known that from electricity theory one of \(V_i\) can take arbitrary value, especially zero.

When \(z \to \infty\), Eqs. (15) and (18) are respectively reduced to

\[\lim_{z \to \infty} F_4(z_4) = B_{44}^{-1} \int_{L^2} D_2^-(x_1)dx_1/2\pi iz_4,\]

\[\lim_{z \to \infty} F_4(z_4) = iB_{44}^{-1}\gamma_{n-1}/2z_4.\]

Therefore we have

\[\gamma_{n-1} = \frac{1}{\pi i} \int_{L^2} D_2^-(x_1)dx_1 = -\frac{1}{\pi} (-Q) = \frac{Q}{\pi}.\]  

\(28\)

It is also noted that if \(E_i\) is determined by using Eq. (26), then the potentials can be determined as follows: Let \(\varphi_1 = V_0\) at electrode 1 \((a_1 \leq x \leq b_1, y = 0)\), then \(\varphi_i\) at electrode \(i (a_i \leq x \leq b_i, y = 0)\) can be determined by

\[\varphi_i = \varphi_{i-1} + \int_{b_{i-1}}^{a_i} E_1 dx_1 = V_i, \quad i = 2, 3, \ldots, n.\]  

\(29\)

So the relations between \(V_i\) and \(q_i\) are obtained. It means that Eqs. (9a) and (9b) are equivalent.

4. Some special cases

4.1.

When the collinear electrodes are degenerated into a single surface electrode with charge \(q\) and \(a_1 = -a, b_1 = a\), then using Eq. (28) the function \(F_2(z_s)\) is

\[F_2(z_s) = B_{44}^{-1} \frac{P(z_s)}{2\sqrt{(z_s^2 - a^2)}} = B_{44}^{-1} \frac{q}{2\pi \sqrt{(z_s^2 - a^2)}}.\]

\(30\)

Integrating Eq. (30) we have

\[f_2(z_s) = \frac{iq}{2\pi} \left\{ \ln \left( z_s + \sqrt{z_s^2 - a^2} \right) + \ln \beta \right\} B_{44}^{-1},\]  

\(31\)
where $\beta$ is a real constant. Let $\varphi = V_0$ on the electrode, then we have

$$
\varphi = 2 \text{Re} \{ A_{44} f_2(z_4) \} = \text{Re} \left\{ i A_{44} B_{44}^{-1} \frac{q}{\pi} \left[ \ln(x_1 - i \sqrt{a^2 - x_1^2}) + \ln \beta \right] \right\} = V_0.
$$

Because $H_{44} = i A_{44} B_{44}^{-1}$ is real and the modulus of $x_1 - i \sqrt{a^2 - x_1^2}$ is $a$, we get

$$
\frac{q}{\pi} H_{44} \ln \beta = V_0 \quad \text{or} \quad \beta = \frac{1}{a} \exp \left( \frac{V_0 \pi}{q H_{44}} \right).
$$

(32)

So in this case the electric potential $\varphi$ is

$$
\varphi = \frac{q H_{44}}{\pi} \text{Re} \left\{ \ln \frac{1}{a} \left( z_4 + \sqrt{z_4^2 - a^2} \right) \right\} + V_0.
$$

(33)

Usually $H_{44} < 0$. Now the normalized stresses can be expressed as

$$
t_{2\beta} = -\frac{q}{\pi} \text{Im} \sum_{x=1}^{4} B_{\beta x} \frac{1}{\sqrt{(z_x^2 - a^2)}} B_{44}^{-1}
$$

$$
t_{1\beta} = \frac{q}{\pi} \text{Im} \sum_{x=1}^{4} B_{\beta x} \frac{p_x}{\sqrt{(z_x^2 - a^2)}} B_{44}^{-1}
$$

(34)

The above equations are identical with Ref. [9]. For the dielectric without piezoelectric effect then $Q, R, T$ each has one component, $Q_{44} = -\varepsilon_{11}, R_{44} = -\varepsilon_{13}, T_{44} = -\varepsilon_{33}$. The eigenvalue equation is reduced to one equation,

$$
-(\varepsilon_{11} + 2p_4\varepsilon_{13} + p_4^2\varepsilon_{33})a_4 = 0, \quad p_4 = p = \left[ -\varepsilon_{13} + i \sqrt{\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2} \right] / \varepsilon_{33}.
$$

(35)

Thus $A, B$ each only has one component, $B_{44} = -(\varepsilon_{12} + p_4\varepsilon_{33})$,

$A_{44} = -i \sqrt{\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2} a_{44}$ and $H_{44} = -(\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2)^{-1/2} < 0$. The results become

$$
F_{44}(z_4) = \left[ \frac{iq}{2\pi \sqrt{z_4^2 - a^2}} \right] B_{44}^{-1},
$$

(36)

$$
\varphi(z_4) = V_0 - \frac{q}{\pi \sqrt{\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2}} \text{Re} \ln \left[ \frac{z_4}{a} + \sqrt{\left( \frac{z_4}{a} \right)^2 - 1} \right].
$$

(37)

For isotropic medium, $\varepsilon_{ij} = \varepsilon \delta_{ij}$, $\varphi = V_0 - \frac{q}{\pi \varepsilon} \text{Re} \ln \left[ z_{a} + \sqrt{(z_{a}^2 - 1)} \right]$, which is identical with the result in the usual dielectric theory.
4.2.

Two surface electrodes with same configuration are considered as shown in Fig. 2.

4.2.1.

Case 1: The electrodes 1 and 2 are located at \( a_1 = -0.03 \) m, \( b_1 = -0.01 \) m, \( a_2 = 0.01 \) m, \( b_2 = 0.03 \) m and the charges on them are \(-q\) and \(q\) respectively. From Eq. (28) it is obvious that \( \gamma_1 = 0 \). \( \gamma_0 \) can be obtained by numerical method and \( \gamma_0 = 1.1469q \). Obviously if let \( \phi_1 = V_0 \) on electrode 1, then \( \phi_2 = V_0 - \int_{-1}^{1} E_1 dx_1 \) on the electrode 2. The problem is solved. By numerical integration we get \( \phi_2 = V_0 + 1.0707q \).

4.2.2.

Case 2: The electric charge of each electrode is \( q \) and their ends are located at \( a_1 = -0.03 \) m, \( b_1 = -0.01 \) m, \( a_2 = 0.01 \) m, \( b_2 = 0.03 \) m, respectively. PZT-5H ceramic with poling axis \( \alpha_2 \) is taken as an example. The material constants of which are listed in Table 1. The function of Eqs. (17) and (28) are respectively reduced to

\[
P(z) = i(\gamma_1 z + \gamma_0), \quad \gamma_1 = \frac{2}{\pi} q.
\]

Due to the symmetry of the distribution of electric charge, so \( \gamma_0 = 0 \).

<table>
<thead>
<tr>
<th>TABLE 1: Elastic stiffnesses, piezoelectric coefficients and dielectric constants of PZT-5H^a</th>
</tr>
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<tbody>
<tr>
<td>( c_{11} ) (N/m(^2))</td>
</tr>
<tr>
<td>--------------------------------</td>
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<tr>
<td>12.6 \times 10^{10}</td>
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</tbody>
</table>

^a The constitutive equations are: \( \sigma_i = c_{ij} e_j - e_{ij} E_j \), \( D_i = e_{ij} e_j + \kappa_{ij} E_j \).
4.2.3. Numerical results: Figs. 3–6 give the distributions of normalized electric displacement $(D_1/q, D_2/q)$ and normalized stresses $(\sigma_{12}/q, \sigma_{22}/q)$ with $x_1/a$ along axis $ox_1$ at different $x_2/a$.

Fig. 3. Distributions of electric displacement $D_2/q$ with $x_1/a$ axis ($x_2/a = -1, -0.1, -0.01$). (a) Case 1; (b) Case 2.
values \( (x_2/a = -1, -0.1, -0.01) \), respectively. As shown in these figures, we can see that in case 1, the stress \( \sigma_{22} \) and electric displacement \( D_2 \) are anti-symmetric to \( x_2/a \), but \( \sigma_{21} \) and \( D_1 \) are
symmetric to $x_2/a$. In case 2, the stress $\sigma_{12}$ and electric displacement $D_1$ are anti-symmetric to $x_2/a$, but $\sigma_{21}$ and $D_1$ are symmetric to $x_2/a$. From Eq. (29) and the anti-symmetry of $E_1$ related to $x_2/a$. 

Fig. 5. Distributions of stress $\sigma_{12}/q$ with $x_1/a$ axis ($x_2/a = -1, -0.1, -0.01$). (a) Case 1; (b) Case 2.
in case 2, the electric potentials on two electrodes are the same. But in case 1 the potentials on two electrodes are different because $E_1$ is symmetric to axis $x_2$. When $x_2/a$ approaches to zero, the
electric and mechanical fields are singular (with inverse square root singularity) at the vicinity of the electrode ends. The distribution behaviors of electroelastic fields at the place between two electrodes \((-1 < x_1/a < 1)\) and at the place outside the electrodes \((x_1/a < -3 \text{ or } x_1/a > 3)\) are different. This is shown that the effects of multiple electrodes are obvious. Comparing the values of two cases we can see that the region between two electrodes and \(x_2\) is not large, then the electroelastic fields in case 1 are larger than that in case 2.

**Fig. 7.** Distributions of stresses in depth direction \((x_1/a = 3, 2, 1, 0.5, 0)\). (a) \(\sigma_{22}/q\); (b) \(\sigma_{12}/q\).
The calculations also show that the values of electric displacement $D_2/q$ on $|x_2/a| < 4$, $x_2/a = -10$ are less than $6 \times 10^{-3}/m^2$ and $5.8 \times 10^{-2}/m^2$ for case 1 and 2 respectively. It is shown that at the remote area from the electrodes, the values of $D_2/q$ approach to zero in both cases. But the speed approaching zero in case 1 is faster than that in case 2. The further study shows that the potential in case 1 is finite at infinity but the material in case 2 approaches $-\text{const.} \ln r$ as $r \to \infty$. 

Fig. 8. Distributions of electric displacements in depth direction ($x_1/a = 3, 2, 1, 0.5, 0$). (a) $D_2/q$; (b) $D_1/q$. 
The behaviors of normalized stresses and normalized electric displacements along $x_2/a$ axis at different $x_1/a$ values shown in Figs. 7 and 8 respectively are only calculated for case 2. As expected, stresses $\sigma_{22}$ and $\sigma_{12}$ vanish on the electrodes and free surface. The electric displacement $D_2$ vanishes on the free surface and has a finite value on the electrodes. The electric field $E_1$ vanishes on the surface electrodes and takes nonzero value on the free surface (which is not shown by figures). At the vicinity of the electrode ends ($x_1/a = -3, -1, 1, 3, x_2/a = 0$), the electric and mechanical fields are concentration, which will induce depoling and even breakdown. The high electric and mechanical fields near the electrode ends are declined quickly when the depth $(-x_2/a)$ increases. At a finite distance from the surface $D_2(E_2)$ approaches a lower even value, $D_1$ and $\sigma_{12}$ approach zero, but $\sigma_{22}$ varies still finite. All these variables approach zero when $|z| \to \infty$.

The situation for case 1 can be discussed in a similar manner.

5. Conclusions

The problem of the half-plane with thin collinear surface electrodes of piezoelectric ceramic is formulated and solved in a closed form. Some special cases with numerical results are given. For these examples, the distributions of electric and mechanical fields are given graphically. These figures show that the electroelastic fields of multi-electrodes are different with that for one electrode. It is also shown that there are significant differences for the case with zero total charge on all electrodes and the case with $Q \neq 0$. It is found that electric and mechanical fields are singular at the vicinity of the electrode ends. These results are important for the function and failure to the piezoelectric apparatuses, especially for the interdigitated electrodes.

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